Non-commutative Quantum Geometry: a re-appraisal of the Bohm approach to quantum theory.

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Abstract.

In this paper we trace the mathematical origins of the wave and particle aspects of quantum phenomena to the symplectic symmetry, *Sp*(2*n*). This symmetry is shared by both classical mechanics and quantum mechanics. We show how the quantum formalism appears in the covering space of the symplectic group. In the quadratic Hamiltonian approximation, it is the covering metaplectic group, *Mp*(2*n*), that gives the Schrödinger equation directly. As is well-known it is the generalisation to all Hamiltonians, to *Ham* (2*n*), that presents mathematical difficulties. Recently de Gosson has shown how to derive the Schrödinger equation even in this case. Our approach is to address the problem through the non-commutative algebraic approach, which has its origin in the work of Emch. We show how the Schrödinger equation and its dual can be expressed in a totally algebraic form that involves both the commutator and the anti-commutator (or Jordon product). We show that the resulting two equations project directly into the equations forming the basis of the Bohm interpretation. In fact we can regard the Bohm approach as providing a way to construct shadow manifolds expected from the ideas of non-commutative geometry. Finally we discuss some of the consequences that follow from this mathematical structure.

1. Introduction.

I want to use this opportunity to review the new developments that have taken place since the appearance of the book "The Undivided Universe" which I completed with David Bohm just before he died. (See Bohm and Hiley 1987, 1993). In some of the literature this approach has been known as the de Broglie-Bohm approach or more lately as Bohmian mechanics. But it should be noted that discussions using these names sometimes differ in important respects from what Bohm and I had in mind when we wrote our book. I do not want to go into these differences here as I would prefer to go directly to the new developments that have emerged from a fusion of my own ideas (Brown and Hiley 2000, Hiley 2001, 2002 & 2003) with those of de Gosson (2001). This work throws a very different light on the formalism first proposed by Bohm (1952). In this paper I would like to summarise the main developments that have occurred and to explain how I now see the Bohm approach.

I want to first start by recalling some of the key steps in Schrödinger's original attempt to derive his equation from classical physics. Of course he failed to provide a mathematically 'clean' derivation as he himself acknowledged in his original paper. However it has been realised for some time now that it is possible to rigorously derive the Schrödinger equation from classical symplectomorphisms by lifting the classical phase space behaviour onto a covering space provided the Hamiltonian is up to quadratic in position and momentum. (See Guillemin and Sternberg 1990 and de Gosson 2001) It this lifting process that enables the wave and particles aspects to be described in a united formalism. In effect the particle properties are described on the underlying phase space while the wave properties appear at the level of the covering space. A summary of these ideas will be discussed in section 2.

The lift onto a more generalised covering space, $Ham(2n)$, has presented considerable mathematical difficulties (see Guillemin and Sternberg 1990). De Gosson (2001) has recently shown how these difficulties can be avoided and the Schrödinger equation can be derived rigorously for all Hamiltonians. In section 3 I will describe how the mathematical difficulties of $Ham(2n)$ can also be avoided by using some of the ideas that are available from non-commutative geometry. This has meant applying the purely algebraic approach to quantum mechanics using the ideas detailed in Emch (1972). In more modern terms this has meant exploiting the properties of the symplectic Clifford algebra and using the less well-known structure of the *symplectic spinor* (See Crumeyrolle, 1990). Because this algebraic structure is non-commutative it is not possible to obtain a unique underlying phase space as one can from a commutative structure. What one is forced to do is to construct so-called shadow manifolds and one such manifold is the Bohm phase space. Here we will see the precise reason for the appearance of the quantum potential, which some physicists regard as *ad hoc*. It turns out that its appearance is a direct consequence of projecting the non-commutative algebraic structure onto a shadow manifold.

In section 4 we show that there are (infinite) many shadow phase spaces that can be constructed, each with their own quantum potential. Thus it is possible to obtain a Bohm approach in the momentum representation, so removing the criticism that the original proposals of Bohm produced an asymmetry that is not in the original formalism. Our work shows that the Bohm approach is deeply embedded in the standard formalism and simply provides an alternative perspective to it. What Bohm does is simply choose the position representation to be special and develops the interpretation on that basis. I believe our work shows that there are no scientific grounds for arguing that the Bohm interpretation is in some way fundamentally flawed. It is simply another way of looking at the same formalism. We conclude by discussing the consequences of this new way of looking at the Bohm interpretation.

2. The Schrödinger equation.

To obtain his equation, Schrödinger (1926) started with time independent classical Hamiltonian–Jacobi equation

$$H(\mathbf{r}, \ S) = E \tag{1}$$

Here S is the classical action. Schrödinger then writes $S = K \ln\psi$ so that the Hamilton-Jacobi equation for a particle in a classical potential, V, becomes

$$(\ \psi)^2 - \frac{2m}{K}(E - V)\psi^2 = 0 \tag{2}$$

To obtain the wave equation from equation (2), Schrödinger assumed that H is a quadratic function in ψ and its derivatives. He then performs the following variation

$$\delta J = \delta \iiint dxdydz \left[(\ \psi)^2 - \frac{2m}{K}(E + V)\psi^2 \right] = 0$$

This immediately gives the Schrödinger equation for a particle with constant energy in a classical potential V. Of course it was not clear even to Schrödinger what this variation means because he writes in a footnote: "I realise that this formulation in not unambiguous" (Schrödinger 1926).

Thus we see historically the Hamilton-Jacobi equation played an important role in leading to the Schrödinger equation. The Hamilton-Jacobi equation also plays a key role in the Bohm interpretation, which we shall discuss in section 3. The key question that I want to raise here is whether we can derive the Schrödinger equation from the classical formalism in a rigorous way. I should remark that this work was partially motivated by Polkinghorne's remarks the this equation "came out of Schrödinger's head" (Polkinghorne 2002).

2.1 Symplectic symmetries of classical mechanics.

To show how we can derive the Schrödinger equation from classical mechanics I must first return to remind ourselves how the symplectic group arises in classical mechanics. We begin with Hamilton's equations of motion, which can be written succinctly in the form

$$\frac{d}{dt}(x, p) = X_H(x, p) \tag{3}$$

where $X_H = \left({}_p H, - {}_x H \right)$. By solving these equations we can write the dynamics in the form

$$(x, p) = f_{t,t_0}(x_0, p_0) \tag{4}$$

where f_{t,t_0} is a symplectomorphisms. Infinitesimal symplectomorphisms $s_{t,t0}$ satisfy the relation

$$\tilde{s} J s = J \text{ where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In practice these can be most easily found, not by solving Hamilton's equation of motion, but by solving the corresponding Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(\mathbf{r}, \quad S) = 0.$$

which for a particle in a classical potential $V(r, t)$ takes the form

$$\frac{\partial S}{\partial t} + \frac{(\quad S)^2}{2m} + V = 0 \tag{5}$$

We now illustrate how this works for a free particle in one dimension. We first solve the Hamilton-Jacobi equation and find

$$S(x, x_0, t, t_0) = \frac{m(x - x_0)^2}{2(t - t_0)} \tag{6}$$

with $p = \frac{\partial S}{\partial x}$ and $p_0 = -\frac{\partial S}{\partial x_0}$. Then it is easy to show that

$$\begin{aligned} x &= x_0 + \left(\frac{t - t_0}{m} \right) p_0 \\ p &= p_0 \end{aligned} \qquad f_{t,t_0} = \begin{pmatrix} 1 & \frac{t - t_0}{m} \\ 0 & 1 \end{pmatrix} \tag{7}$$

We can also easily verify that f_{t,t_0} is a symplectic matrix.

We can carry through the same procedure for the one-dimensional harmonic oscillator using the function

$$S_{HO} = \frac{m\omega}{2 \sin \omega t} \left[\left(x^2 + x_0^2 \right) \cos \omega t - 2 x x_0 \right] \tag{8}$$

to obtain the well-known equation of motion. In this way we see how $S(\boldsymbol{r},\boldsymbol{r}_0,t,t_0)$ generates the classical motion.

2.2 Ray optics.

In order to see why the symplectic group plays a key role in the derivation of a wave equation from particle mechanics let me briefly recall how geometric optics is also based on this group structure. Recall Fermat's principle in which the equation of a light ray can be derived from the variation of the optical path

$$\int n d\sigma = \int n\sqrt{1+\dot{x}^2+\dot{y}^2}\,dz = \int L dz \qquad (9)$$

where we have written

$$d\sigma = \sqrt{dx^2+dy^2+dz^2} = \left(\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dx}{dz}\right)^2}\right)dz = \sqrt{1+\dot{x}^2+\dot{y}^2}\,dz$$

so that

$$L = \sqrt{1+\dot{x}^2+\dot{y}^2}\,dz$$

Obtaining the conjugate momenta, $p = \frac{dL}{dz}$, from L we find

$$p_x = n\frac{\dot{x}}{\sqrt{1+\dot{x}^2+\dot{y}^2}} \qquad p_y = n\frac{\dot{y}}{\sqrt{1+\dot{x}^2+\dot{y}^2}}$$

so that the equivalent Hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - L = -\sqrt{n^2 - p_x^2 - p_y^2} \qquad (10)$$

Hamilton's equations of motion are then

$$\dot{x} = \frac{\partial H}{\partial p_x}; \qquad \dot{p}_x = -\frac{\partial H}{\partial x} \qquad \dot{y} = \frac{\partial H}{\partial p_y}; \qquad \dot{p}_y = -\frac{\partial H}{\partial y}$$

The equation corresponding to the Hamilton-Jacobi equation (1) is then

$$\frac{\partial S}{\partial z} - \sqrt{n^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2} = 0$$

$$\left(\nabla S\right)^2 - n^2 = 0 \qquad (11)$$

which is just the well-known eikonal equation.

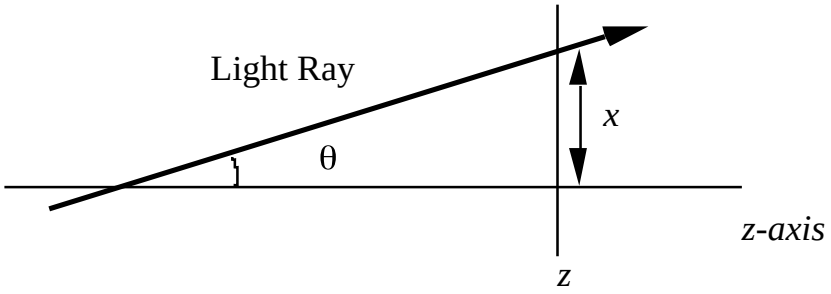

Figure 1. Co-ordinates (x, $p = n\theta$) of light ray.

To identify the co-ordinates (x, p_x) consider a ray that travels in the $x - z$ plane, then x is the height of the ray above the z-axis at the point z. $p = n\theta$, where θ is the angle the ray makes with the z-axis. (See figure 1.)

In terms of these co-ordinates the equation of the ray can also be written as a symplectomorphism, which we can write in the form

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \tag{12}$$

Then for a ray travelling in uniform medium of refractive index n we have

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \qquad d = \frac{\Delta t}{n}. \tag{13}$$

While for a ray passing through a boundary $n_1 \rightarrow n_2$ we have

$$\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix}, \qquad P = \frac{(n_1 - n_2)}{R}.$$

Then for a lens, the symplectomorphism is given by

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -u/f & u + v - uv/f \\ -1/f & 1 - v/f \end{pmatrix}$$

If planes are conjugate then upper off-diagonal element is zero. From this we can find the thin lens formula

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

I derive it in this manner just to stress that underlying ray optics and classical mechanics is the symplectic group. I now want to make use of this fact to motivate the notion of a covering space, which we will use to describe the wave properties generated by both types of processes.

2.3 Waves and rays from Huygens's Construction.

Let us now recall how a series of wave fronts is related to the eikonal equation. Consider a wave front given by the equation $ct = S(x,y,z)$ so that

$$cdt = dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz \qquad (14)$$

In figure 2 let the ray P_0P be described by direction cosines $\alpha = \frac{\partial x}{\partial \sigma}, \quad \beta = \frac{\partial y}{\partial \sigma}, \quad \gamma = \frac{\partial z}{\partial \sigma}$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$so that

$$N\alpha = \frac{\partial S}{\partial x}, \quad N\beta = \frac{\partial S}{\partial y}, \quad N\gamma = \frac{\partial S}{\partial z}$$

Substituting into equation (14) we find

$$cdt = N(\alpha dx + \beta dy + \gamma dz) = N\left(\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial y}{\partial \sigma}\right)^2 + \left(\frac{\partial z}{\partial \sigma}\right)^2\right)^{\frac{1}{2}} d\sigma = Nd\sigma$$

Now if the light has velocity v in a refractive medium then $d\sigma = \mathrm{v}dt$ and $n = \frac{c}{\mathrm{v}}$ so that $cdt = nd\sigma$ and $N = n$. Since $\alpha^2 + \beta^2 + \gamma^2 = 1$ we have,

$$\frac{1}{n^2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{n^2}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{n^2}\left(\frac{\partial S}{\partial z}\right)^2 = 1$$

which is just the eikonal equation (11)

2.4 The wave equation.

In the optical case considered in this sections we begin to see how rays and waves are related. What remains to be done is to describe how the phase changes with time. To begin with note that the solution of the eikonal equation is

$$S(P_0, P) = \frac{n}{2}\frac{(x - x_0)^2}{(t - t_0)} \qquad (15)$$

which should be compared with the solution of the Hamilton-Jacobi equation for the free particle given in equation (6). Here this is simply the optical path length between

P_0 and P. This will produce the phase change $\exp[2\pi ikS(P_0, P)]$ where $k = 1/\lambda$, λ being the wave length of the light. As the light travels along its path, the amplitude decrease by some factor depending on the length of the path so that the wave will be modulated by

$$A(P_0, P)\exp[2\pi ikS(P_0, P)]$$

If initially we have the wave $\psi_0(x_0)$ then the contribution to the wave at x from the secondary source at x_0 will be

$$\psi(x) = A(x, x_0)\exp[2\pi ikS(x, x_0)]\psi_0(x_0)$$

If we now assume that the Huygens construction is valid, we get

$$\psi(x) = \int A(x, x_0)\exp[2\pi ikS(x, x_0)]\psi_0(x_0)dx_0 \qquad (16)$$

[NB I have shortened the analysis and there is a slight complication, which is fully discussed in Guillemin and Sternberg (1990). This does not affect our conclusions.]

Feynman (1948) shows is that equation (16) is exact for free particles provided we use
$S(x, x_0)$as defined in equation (6). Does this mean that we are returning to the Feynman path integral approach? Not according to de Gosson (2001)! He argues that provided we accept the de Broglie matter wave hypothesis, the Schrödinger equation emerges from classical mechanics not only for quadratic Hamiltonians, but for *all* Hamiltonians. Furthermore the emergence is mathematically exact. To explain his argument here would take us too far from the main point I am making in this paper so I will refer the interested reader to his excellent book. For the purposes of this paper the point I want to bring out is the connection of this work to the Bohm interpretation.

3. Covering groups.

3.1 The symplectic and metaplectic groups.

From the mathematical point of view what the Huygens construction has done is to lift the Hamiltonian flow from the classical phase space onto the covering space. For quadratic Hamiltons this generates the metaplectic group, which is the double cover of the symplectic group. (For more details see Guillemin and Sternberg 1978, 1990 and de Gosson 2001).

For the purposes of this paper let us define an operator U_s by

$$\left(U_s\psi\right)(x) = A\int \exp\left[2\pi ikS(x,x_0)\right]\psi_0(x_0)dx_0 \qquad (17)$$

Then obviously

$$\left(\left(U_{S_1}\circ U_{S_2}\right)\psi\right)(x) = A\int \exp\left\{2\pi ik\left[S_1(x,x_0)+S_2\left(x,x_0\right)\right]\right\}\psi_0(x_0)dx_0$$

so that

$$U_{S_1}\circ U_{S_2} = U_{S_1+S_2} \qquad (18)$$

Thus the covering clearly has a group structure. If we take the special case where $S(x, x_0)$ is defined by

$$S(x,x_0) = \frac{1}{2B}\left[Dx^2 - 2xx_0 + Ax_0^2\right] \qquad (19)$$

we find that that this generates the metaplectic group, $Mp(2n)$ which double covers the symplectic group generated by

$$s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where in $2n$ dimensional phase space. The matrices A, B, C and D satisfy the relations:-

$$\tilde{A}C, \tilde{D}B \text{ sym. } \tilde{A}D - \tilde{C}B = 1\text{: } A\tilde{B}, C\tilde{D} \text{ sym. } A\tilde{D} - B\tilde{C} = 1\text{: and } D\tilde{C}, A\tilde{B} \text{ sym. } D\tilde{A} - C\tilde{B} = 1$$

3.2 Schrödinger equation.

One can find in the metaplectic group families of operators $U(t)$ $Mp(2n)$ that form a one-parameter subgroup satisfying $U(t_1)\,U(t_2) = U(t_1+t_2)$. Corresponding to this one-parameter group there will be a one-parameter group of matrices $M(t)$ $Sp(2n)$.

Since M depends on t we can always write $\frac{dM(0)}{dt} = K$ so that $M(t) = e^{tK}$.

Now for small $|t|$ we can uniquely recover $U(t)$ from $M(t)$. Once we know $U(t)$ for small $|t|$ we know it for all t because $U(t) = U(t/n)^n$. Since dU/dt is a skew adjoint operator, we can write $dU/dt = -iH$, where H is a self-adjoint operator. Thus $U(t)$ satisfies Schrödinger type equation

$$\frac{dU}{dt} = -iHU \qquad (20)$$

By examining the details of the matrix f_{t,t_0} given in equation (7) it is not difficult to show that t is time. This approach is exact for Hamiltonians that are up to quadratic in x and p. This is clearly shown in equation (19).

Before going on to explain how to generalise this approach to all Hamiltonians, let us just display the connection more explicitly for the free particle and the one-dimensional harmonic oscillator.

In classical mechanics we have the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(\boldsymbol{r}, \quad S) = 0$$

The solution of this equation $S(\boldsymbol{r}, \boldsymbol{r}_0, t, t_0)$ generates the classical motion. Thus for the free particle we find

$$S_{FP} = \frac{m(\boldsymbol{r} - \boldsymbol{r}_0)^2}{t - t_0} \tag{21}$$

While the generating function for the harmonic oscillator is

$$S_{HO} = \frac{m\omega}{2\sin\omega t}\left[\left(x^2 + x_0^2\right)\cos\omega t - 2xx_0\right] \tag{22}$$

Both of these are special cases of the quadratic expression shown in equation (19).

Now let us compare these equations with the solutions of the Schrödinger equation (20), which we now write in the more familiar form

$$i\frac{\partial \psi}{\partial t} = H\psi$$

The Green's function solutions, $G(\boldsymbol{r}, \boldsymbol{r}_0, t, t_0)$, of Schrödinger's equation propagates the quantum motion. For the free particle the Green's function is

$$G_{FP} \quad \exp\left[\frac{m(\boldsymbol{r} - \boldsymbol{r}_0)^2}{t - t_0}\right] = \exp\left[iS_{FP}(\boldsymbol{r}, \boldsymbol{r}_0)\right] \tag{23}$$

While the harmonic oscillator gives

$$G_{HO} \quad \exp\left[\frac{im\omega}{2\sin\omega t}\left[\left(x^2 - x_0^2\right)\cos\omega t - 2xx_0\right]\right] = \exp\left[iS_{HO}(\boldsymbol{r}, \boldsymbol{r}_0)\right] \tag{24}$$

Note here that the Green's functions are just the lift of the classical Hamiltonian flow and are clearly directly related to the representation of the operators in the metaplectic group $Mp(2n)$. For convenience we present an image of how the classical phase space is covered by the space in which the metaplectic group operates in figure 2.

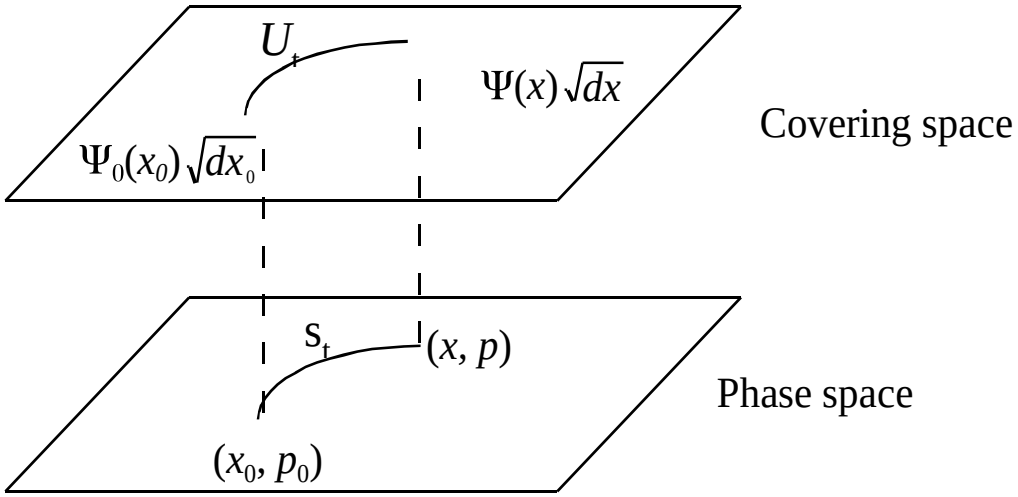

Figure 2: The relation between the classical phase space and the covering space of quantum mechanics.

3.2 Symplectic spinors.

The object that takes the place of the co-ordinates (x, p) of classical phase space is the symplectic spinor $\psi(x)\sqrt{dx}$. This is also known as a half-form. It is simply the symplectic analogue of the Pauli spinor. The root of dx appears because of the normalisation condition that holds both in electromagnetism and in quantum mechanics. That is we must have

$$\int |\psi(x)|^2 dx = \int |\psi_0(x_0)|^2 dx_0 \qquad (25)$$

Thus equation (16) should be written in the form

$$\psi(x)\sqrt{dx} = A\int \exp\left[2\pi i k S(x, x_0)\right]\psi_0(x_0)\sqrt{dx_0} \qquad (26)$$

If we substitute equation (26) in the normalisation condition (25) we find that $A = e^{\mp i\pi/4}|\det \lambda B|^{-1/2}$. With this result we can now show how the double cover arises.

To this end first notice that when $A = 0$, $B = 1$, $C = -1$, $D = 0$ the transformation becomes the Fourier transform $(x, p) = F(x_0, p_0)$. This corresponds to a rotation through $\pi/2$ in the phase space. Thus

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \quad \text{so that} \quad x_0 \to -p \quad \text{and} \quad p_0 \to x$$

Clearly we need to apply the Fourier transformation four times to return to our starting point (that is, four lots of π/2 gives 2π!) Thus since $F(Ff)(x) = f(-x)$ we get $F^4 = 1$.

Now let us see what happens when we go to the double cover. The metaplectic group element corresponding to the Fourier transformation on the phase space is

$$U_F\psi(x) = e^{-\pi i/4}|k|^{1/2}\int \exp[-2\pi ikxx_0]\psi_0(x_0)dx_0 \qquad (27)$$

so that $U_F = e^{-\pi i/4}F$. This gives immediately

$$\left(U_F\right)^4 = \left(e^{-i\pi/4}F\right)^4 = -F \qquad (28)$$

This means that we have to go through another four operations with the group element (26) to get back to where we started. This is a well-known property of any double cover, which has physical consequences demonstrated by the Guoy effect (see Carpenter 1959)

3.4 Heisenberg Algebra and the Metaplectic Group.

In this sub-section I will show how these ideas are contained in the Heisenberg group and its corresponding Lie algebra, which can also provide an alternative and perhaps more formal way of showing the equivalence of the Schrödinger and Heisenberg pictures which we will exploit later in the paper.

Recall that the Heisenberg algebra is generated by $\{1, Q, P\}$ with $[Q, P] = i$. It is not difficult to show that the enveloping algebra of Heisenberg algebra is part of a larger structure known in the mathematics literature as the symplectic Clifford algebra (See Bacry and Boon 1987, and Crumeyrolle 1990). The reason for this name is that it is the symplectic analogue of the more familiar orthogonal Clifford algebra, which is central to the Dirac theory of the electron. Just as there is a double cover of the orthogonal group which give rise to the well known Pauli and Dirac spinors, the symplectic group has a double cover known as the metaplectic group which in turn gives rise to symplectic spinors discussed in the previous section. It is these symplectic spinors that play a central role in this paper.

The corresponding group, the Heisenberg group, is spanned by the elements

$$H(\alpha,\beta,\varepsilon) = \exp\left[\alpha Q + \beta P + \varepsilon Z\right] \qquad (Z \quad) \qquad (29)$$

while the Metaplectic group (the corresponding Clifford group of the symplectic structure) is spanned by the elements

$$M(\alpha,\beta,\varepsilon) = \exp[\alpha Q^2 + \beta P^2 + \varepsilon(QP + PQ)] \qquad (30)$$

The infinitesimal generators of this group are

$$L_0 = 1/4\left[Q^2 + P^2\right]; \quad L_1 = -1/4\left[QP + QP\right]; \quad L_2 = -1/4\left[Q^2 - P^2\right]$$

producing a Lie algebra

$$\left[L_0, L_1\right] = iL_2; \quad \left[L_1, L_2\right] = -iL_0; \quad \left[L_2, L_0\right] = iL_1$$

Clearly this is isomorphic to the symplectic Lie algebra Sp(4). As I have shown elsewhere we can extend the symplectic Clifford algebra so that it contains primitive idempotents (Hiley 2001a). Then it is possible to construct the minimal ideals which are the algebraic equivalent of the symplectic spinors of the metaplectic group as we will show in section 5. We will also discuss the relevance of the algebraic spinors in section 5.

4. Quantum mechanics and the Bohm approach.

4.1 Summary of the Bohm approach.

We now want to consider what all this has to do with the Bohm (1952) approach. (For a more up to date account of this approach see Bohm and Hiley 1993 and Holland, 1993). First recall that the two equations defining the Bohm approach emerges from the Schrödinger equation by simply writing the wave function in polar form $\psi(r,t) = R(r,t)\exp[iS(r,t)]$. Then the resulting equation is split into its real and imaginary parts and we find

$$i\frac{\partial P}{\partial t} + \quad (jP) = 0 \qquad (31)$$

which gives us a conservation of probability equation. The real part of the Schrödinger equation gives

$$\frac{\partial S}{\partial t} + \frac{(\quad S)^2}{2m} - \frac{1}{2m}\frac{^2R}{R} + V = 0 \qquad (32)$$

This equation resembles the Hamilton-Jacobi equation except it contains an extra term $Q = \frac{1}{2m}\frac{^2R}{R}$, which has been called the quantum potential since it is this term that distinguishes classical mechanics from quantum mechanics. If we identify S with

the momentum and regard Q as a new quality of energy only playing a role in quantum processes, then we can regard equation (32) as an expression of the conservation of energy.

It is this term that has provoked some hostile criticism to the Bohm approach. Heisenberg (1958) himself called it *ad hoc*, a sentiment that is still repeated in Polkinghorne (2002). Yet there is nothing *ad hoc* about it. It is a direct consequence of simply rewriting the Schrödinger equation in its real and imaginary parts under polar decomposition of the wave function. Even Dürr, Goldstein and Zanghi (1996) who are strong supporters of 'Bohmian mechanics' seem to find "a serious flaw in the quantum potential approach" but I have failed to understand their specific objections particularly as they argue that the approach is "completely defined by the Schrödinger equation".

I find the fact that equation (32) emerges from the Schrödinger equation far from surprising given the history of the origin of this equation. Indeed we can see even more clearly how the quantum potential is a necessary feature of the description if we follow the suggestion of de Gosson (1998, 2001) and introduce what he calls the Bohmian defined by

$$H^{\psi} = H + Q^{\psi}$$

If this is inserted into the Hamilton-Jacobi equation written in the form

$$\frac{\partial S}{\partial t} + H^{\psi}\left(\boldsymbol{r}, \nabla_{\boldsymbol{r}} S\right) = 0 \qquad (33)$$

then one can show that there exists a symplectomorphism f^{ψ}_{t,t_0} given by $\left(\boldsymbol{r}^{\psi}(t), p^{\psi}(t)\right) = f^{\psi}_{t,t_0}\left(\boldsymbol{r}_0, p_0\right)$, which can be written in the form

$$\left(\frac{d\boldsymbol{r}^{\psi}}{dt}\right) = \nabla_{p} H^{\psi} \qquad (34)$$

and

$$\left(\frac{dp^{\psi}}{dt}\right) = -\nabla_{\boldsymbol{r}} H^{\psi} = -\nabla_{\boldsymbol{r}}\left(V + Q^{\psi}\right) \qquad (35)$$

Equation (34) is simply the guidance condition $p = \nabla S$ from which the trajectories are calculated written in an unusual form. Equation (35) is the generalisation of Newton's equation of motion where the classical potential is supplemented with the quantum potential. This equation ensures that the momentum is always conserved.

If we write

$$S\left(\boldsymbol{r}^{\psi}(t), t\right) = S_0(x_0) + \int_0^t \left(\boldsymbol{p}.d\boldsymbol{r} - H^{\psi}\, dt'\right)$$

then it is not difficult to show that

$$\psi\left(\boldsymbol{r}^{\psi}(t),t\right)\left|\, d^n\boldsymbol{r}^{\psi}\,\right|^{1/2} = \exp\left[\frac{i}{\hbar}S\left(\boldsymbol{r}^{\psi}(t),t\right)\right]\psi_0(x_0)\left|\, d^n x_0\,\right|^{1/2} \qquad (36)$$

where $\psi\left(\boldsymbol{r}^{\psi}(t),t\right)$ is a solution of Schrödinger's equation. For proof of these results see de Gosson (1998). This demonstrates the key role the quantum potential plays in the relation between the symplectic space and its double cover as we will elaborate further below.

Now we are in a position to clarify the relation between the phase space and its covering space. We can regard the Schrödinger equation as describing the time evolution of the flow in the quantum system in the covering space. Then for each initial point of the covering space we can project a distribution of initial points on the phase space lying immediately below it. As the Schrödinger equation develops a series of trajectories unfold in the underlying manifold. These determine classical-like symplectic flows, which is defined by the Bohmian. Thus we see the quantum potential plays an essential role in the mathematical relation between the phase space and the corresponding covering space. Furthermore it is the property of covering spaces that ensures that the underlying trajectories do not cross thus explaining a well-known property of the Bohm trajectories. Thus there is nothing *ad hoc* or artificial about the Bohm flows. They are a necessary structure of the quantum formalism when looked at in terms of the underlying geometry. We will look at this result in a new way in the next section.

Before moving on I would like to consider several features of the Bohm approach that are sometimes considered to be 'unsatisfactory'. Firstly the Bohm approach seems to depend heavily on the Schrödinger picture and totally ignores the Heisenberg picture. Yet the claim is that the Bohm approach uses only the standard formalism. Where then are things like the quantum potential 'hiding' in the Heisenberg picture? It must be there somewhere simply because the matrix mechanics gives a mathematically equivalent description, at least for a finite number of degrees of freedom. We will examine this question in the next sub-section.

A further question, which essentially has it roots in the first question, asks why the position representation is taken as basic. It seems as if it completely ignores the complementary momentum representation. Bohm (1953) did address this question and concluded that one could not build a satisfactory momentum story because of problems with the mathematics, but this is not so as we will bring out later.

The third difficulty is to do with the uncertainty principle. How is it possible to build a phase space in terms of (x, p) and yet satisfy the uncertainty principle and avoid the

no-go theorems of Gleason (1957), Kochen and Specker (1967)? This has been philosophically resolved by arguing for context dependence. But there is a very simple mathematical reason why there is no problem. The momentum used in Bohm's approach using the *x*-representation is not the eigenvalue of the momentum operator. It turns out to be only the real part of the momentum operator as we show in equation (55). Thus what Bohm is saying is that the variables (beables) associated with the particle are not the simultaneous eigenvalues of the position and momentum operators. Yes the *x*-coordinate is the eigenvalue of the position operator but the Bohm momentum is not the eigenvalue of the momentum operator. The eigenvalues are what you find when you measure a quantity. Measurement is participatory and that is what makes the context dependence inevitable. Let me try to bring out the mathematical aspect of these things and show how they are related to the non-commutative structure of the quantum formalism.

4.2 Non-commutative algebra and phase space.

I will try to bring out the ideas in a simple way. In classical mechanics position and momentum are always well defined so that the concept of a trajectory in phase space presents no difficulty. Thus the picture shown in figure 3 is unremarkable.

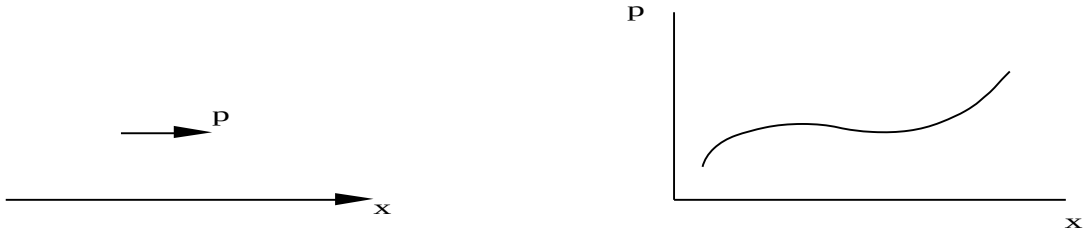

Figure 3

When we come to quantum mechanics we have serious problems if we assume the properties of particles are *only* described by the eigenvalues of operators. In the case of the *x*-representation, we can say the particle has a well-defined position, but its momentum is totally unknown. On the other hand in the *p*-representation the momentum is exactly defined but the position is totally unknown. The appropriate representation is defined by the context and thus the context determines which operator is diagonal. In a position measurement we must diagonalise the position operator. Thus for a simple one-dimensional discrete space, we diagonalise the position operator thus

$$S_1 X S_1^{-1} = X_{dia} = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{pmatrix} \qquad (37)$$

We then find the particle in one of the boxes in figure 4. We can say nothing about its momentum.

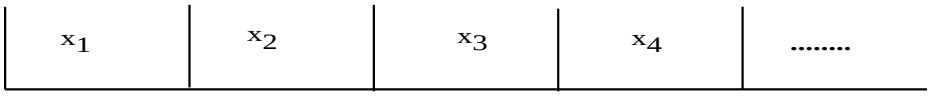

Figure 4

On the other hand if we perform a momentum measurement we must diagonalise the momentum operator

$$S_2 P S_2^{-1} = P_{dia} = \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \end{pmatrix} \qquad (38)$$

This means that we can only represent the particle in one of the *p*-boxes as shown in figure 5, but can say nothing about where it is.

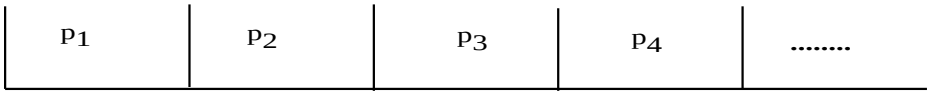

Figure 5

Because of non-commutation $S_1 \neq S_2$ so we can never represent the particle on a phase space defined in terms of the eigenvalues of the *X* and *P*-operators. This is essentially the implications of the work of Kochen and Specker (1967).

The problem arises from the fact that we have given primary physical relevance to the eigenvalues and not the operator algebra. Rather than using eigenvalues we should let the algebra determine the phase space. We should not start with something we cannot observe (i.e., the phase space) and then build a structure up on that. The key question is thus, "Is it possible to start with the algebra of functions and then deduce the properties of the underlying space if indeed there is one?

4.3 Construction of phase spaces.

The traditional way of building a theory is to start with a topological or a metric underlying space and then to construct on it functions that can form an algebra on that space. However Gel'fand has shown us that it is possible to start from a commutative algebra of functions and then to abstract any underlying space. (For details see Demaret *et al* 1997). For example if we have a commutative algebra of regular functions, $C^{\infty}(M)$, then the underlying structure is the affine space over the complex

numbers. In this space the points are the maximal ideals of the algebra $C^{\infty}(M)$. If the algebra is a commutative C* algebra then we can abstract out a compact topological space, and so on.

The algebraic structure of observables in quantum mechanics therefore seems to offer the possibility of supplying a way to construct an underlying phase space. The problem is that the algebra of observables is non-commutative. With a non-commutative algebra things are not as straight forward as the commutative case. It is not possible to find a unique underlying space. The best we can do is to constructing 'shadow' spaces. Indeed this is just what the above discrete structure indicates. But here we have only presented 'half' the space. We have no way of representing the complementary property. Thus we can either construct a position space in which there is no momentum specified or one could specify a momentum space in which case there is no position defined. But there is more that we could do. We can actually construct shadow phase spaces of which the Bohm approach provides one such space. In the rest of this talk I will show exactly how we can construct such a phase space starting from the algebra.

5. The Algebraic Approach.

5.1 The extended Heisenberg algebra

We have already seen in the previous section that quantum mechanics 'lives' in the covering space of symplectomorphisms. We now argue that a more general way to look at this covering space is in terms of the Heisenberg algebra or at least a generalisation of it. The equation of motion in this space is normally taken to be the Heisenberg equation

$$i\frac{d\hat{A}}{dt} + \left[\hat{H}, \hat{A}\right] = 0 \qquad (39)$$

However we will find this is not the way to proceed. We are not simply interested in looking at the time development of the operators because we are still left with the wave function, $\psi(t_0)$, 'frozen' in time and not part of the algebra. We want to retain the time development in the wave function but we need to somehow 'put' the wave function into the algebra. To do this we must go to a more general description of the notion of a state. The way to do this is to use the density operator even in the case of a pure state. For the purposes of this paper it is not necessary to consider mixed states, nevertheless the density operator will be our primary concern.

Now we know that the density operator can be represented in a Hilbert space by

$$\hat{\rho} = |\psi\rangle\langle\psi| \tag{40}$$

But we do not want to use the properties of Hilbert space at present. This allows us the generalisation we need. In fact we will follow Emch (1972) and write the density operator as

$$\hat{\rho} = \hat{\psi}_L \hat{\psi}_R \tag{41}$$

Where $\hat{\psi}_L$ $\left(\hat{\psi}_R\right)$ is an element of the left (right) ideal in the algebra. In the case of the orthogonal Clifford algebra one can construct these ideals with comparative ease as has been shown in Hiley (2003). What we need to do is to find a primitive idempotent in the algebra and then to build the left (right) ideals by simply multiplying from the left (right) using elements of the algebra.

When we come to the symplectic Clifford (Heisenberg) algebra we have a problem. The Heisenberg algebra is nilpotent, which means that the algebra does not contain any idempotents. This does not present us with an insurmountable problem since one can extend the algebra as shown in Frescura and Hiley (1984) and Hiley (2003). But Dirac (1947) already anticipated the need for this generalisation. In effect he added an idempotent to the Heisenberg algebra when he introduced the notion of a standard ket and standard bra. These two objects are relatively unfamiliar object and are not the usual bra and ket used in everyday physics. Nevertheless they are related.

Dirac's idea was essentially to lift the ordinary ket into the algebra, which he did by noting that there is no need to write the original 'bar' in front of the angle bracket. Thus $|\ \rangle$ is replaced by $\rangle$. It is the latter symbol that Dirac calls the standard ket. This enabled him to construct left ideals in the algebra symbolically simply because there is no meaning to multiply on the right. Conversely he replaced $\langle\ |$ with $\langle$ to enable the right ideals to be generated. If we put the standard ket together with the standard bra, $\langle$, we have in effect introduced an idempotent. [For an alternative use of these idempotents see Kauffman (2002 and 2002a)]Thus we can write $\rangle\langle = \varepsilon$. Then $\varepsilon^2 = \varepsilon$ establishing that it is an idempotent. Hence we can write

$$\rho = |\psi\rangle\langle\psi| \quad \hat{\psi}\rangle\langle\hat{\psi} \quad \hat{A}\varepsilon\hat{B} = \hat{\psi}_L\hat{\psi}_R \tag{42}$$

where $\hat{\psi}_L = \hat{A}\varepsilon$ is an element of the left ideal. This is the operator equivalent to the wave function. In the same way $\hat{\psi}_R = \varepsilon\hat{B}$ is an element of the right ideal, which is the operator equivalent to the conjugate wave function. The left ideal $\hat{\psi}_L$ is also called

the algebraic symplectic spinor, while the right ideal $\hat{\psi}_R$ is known as the dual symplectic spinor. These objects are representation free.

In this approach the Heisenberg equation of motion is replaced by two *operator* Schrödinger equations,

$$i\frac{\partial\hat{\psi}_L}{\partial t} = \hat{H}\hat{\psi}_L \qquad \text{and} \qquad -i\frac{\partial\hat{\psi}_R}{\partial t} = \hat{\psi}_R\hat{H} \tag{43}$$

Let me stress again that these equations are representation free. Now if we take the difference between these two equations we get

$$i\frac{\partial\hat{\rho}}{\partial t} + \left[\hat{\rho},\hat{H}\right]_- = 0 \tag{44}$$

This will be recognised as the Liouville equation but written in terms of operators. It can be regarded as an equation governing the time evolution of the amplitude of the process.

The sum of the two equations in (43) gives

$$i\left(\frac{\partial\hat{\psi}_L}{\partial t}\hat{\psi}_R - \hat{\psi}_L\frac{\partial\hat{\psi}_R}{\partial t}\right) = \left[\hat{\rho},\hat{H}\right]_+ \tag{45}$$

Thus we see that we need both the commutator and the anticommutator to give a complete description of the content of the Schrödinger equation and its dual. An interesting argument as to why both equations are needed is given in Hiley (2001b).

We can considerably simplify this equation if we can polar decompose $\hat{\psi}_L$ and $\hat{\psi}_R$ so that we can write

$$\hat{\psi}_L = \hat{R}\hat{U} \qquad \text{and} \qquad \hat{\psi}_R = \hat{U}^\dagger\hat{R}$$

where $\hat{R}$ is positive definite, $\hat{U}$ is unitary. Then we find

$$i\left(\frac{\partial\hat{\psi}_L}{\partial t}\hat{\psi}_R - \hat{\psi}_L\frac{\partial\hat{\psi}_R}{\partial t}\right) = i\hat{R}\frac{\partial\hat{U}}{\partial t}\hat{U}^\dagger - \hat{U}\frac{\partial\hat{U}^\dagger}{\partial t}\hat{R} = \left[\hat{\rho},\hat{H}\right]_+ \tag{46}$$

Now suppose we write $\hat{U} = e^{i\hat{S}}$ where $\hat{S} = \hat{S}^\dagger$

$$\hat{R}\frac{\partial\hat{S}}{\partial t}\hat{R} + \tfrac{1}{2}\left[\hat{\rho},\hat{H}\right]_+ = 0 \tag{47}$$

If we further assume that $\left[\hat{R}, \frac{\partial \hat{S}}{\partial t}\right] = 0$ then this equation simplifies to

$$\hat{\rho}\left(\frac{\partial \hat{S}}{\partial t} + \tfrac{1}{2}\left[\hat{\rho}, \hat{H}\right]_+ = 0 \right. \tag{48}$$

We see that this is simply an equation that describes time development of the phase operator. Thus we see that the two Schrödinger type equations (43) are replaced by the two equations

$$i\frac{\partial \hat{\rho}}{\partial t} + \left[\hat{\rho}, \hat{H}\right]_- = 0 \tag{44}$$

and

$$\hat{\rho}\left(\frac{\partial \hat{S}}{\partial t} + \tfrac{1}{2}\left[\hat{\rho}, \hat{H}\right]_+ = 0 \right. \tag{48}$$

Notice that equation (44) uses the commutator while equation (48) is expressed in terms of the anti-commutator (or Jordon product). These equations were originally derived in Brown and Hiley (2000).

5.2 Relation to the Bohm approach.

I now want to relate this algebraic approach to the Bohm approach. To bring out this connection first note the similarity between the pair of defining equations (44) and (48) and the pair of equations that form the basis of the Bohm approach equations (31) and (32). The first difference is that equations (44) and (48) are operator equations and as we have remarked earlier they are representation free. On the other hand equations (31) and (32) are tied to a specific representation, namely, the position representation. In fact as we show below that equations (44) and (48) produce the two Bohm equations if we go to the *x*-representation. This may be surprising since equation (32) contains the quantum potential whereas equation (48) has nothing that remotely looks like a quantum potential.

In order to show why these two equations contain the same information we need to project equations (44) and (48) into a specific representation. We will first do this in a general representation defined by

$$\hat{A}\,|a\rangle = a\,|a\rangle \tag{49}$$

Then immediately we find that (44) becomes

$$i\frac{\partial P(a)}{\partial t} - \left\langle \left[\hat{H},\hat{\rho}\right]_{-}\right\rangle_{a} = 0 \qquad (50)$$

where $P(a)$ is the probability of finding the particle in $|a\rangle$. Thus (50) is just the Liouville equation, which expresses the conservation of probability. This can be easily shown if we choose the Hamiltonian to be $H = p^2/2m + V$ and replace a by x, i.e., we choose the x-representation. In this representation equation (44) becomes identical to the Bohm equation (31).

In a general representation equation (48) becomes

$$P(a)\frac{\partial S(a)}{\partial t} + \tfrac{1}{2}\left\langle \left[\hat{H},\hat{\rho}\right]_{+}\right\rangle_{a} = 0 \qquad (51)$$

which is clearly an equation for the time development of phase equation thus justifying our original claim .

Equation (51) looks remarkably similar to equation (32) but again it seems as if the quantum potential is still missing. However the quantum potential is actually implicit in the anticommutator. To bring this out we have to choose a particular Hamiltonian. For reasons that will become clear, this time we will choose the Hamiltonian for the harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{Kx^2}{2}$$

Substituting this in equation (51) using the x-representation gives

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x}\left(\frac{\partial^2 R^2}{\partial x^2}\right) = 0 \qquad (52)$$

We immediately see that the quantum potential has appeared giving us an expression for the conservation of energy. If we do the same thing in the p-representation we find

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}\left(\frac{\partial S_p}{\partial p}\right)^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0 \qquad (53)$$

Again we see that the appearance of a quantum potential. Thus we can construct a Bohm interpretation in the p-representation as well. Here equation (53) is again an expression for the conservation of energy. We can simply check this by considering

the ground state of the harmonic oscillator and showing the both equations (52) and (53) give the well-known result $E = \omega/2$.

All this shows is that we can construct a Bohm interpretation for any representation whatsoever. In fact mathematically we have not lost the $x-p$ symmetry that Heisenberg (1958) and others complain about. The full symplectic symmetry is still there and shows that the singling out of the x-representation is not made on mathematical grounds. It is made on other grounds. Indeed as Bohm remarked to me once that it seemed more natural to choose the x-representation because all our experiences are in space-time. I would add to this and argue that our measurement instruments separate different results by positions of pointers etc. Even digital displays of instrument readings are ultimately displayed in space-time. Thus in practice we use space-time to make manifest the physical properties of any system. Further details of this approach appear in Hiley (2002)

5.3 Detail comparisons.

In order to interpret equation (48) as a conservation equation not only do we have to identify Q_x with a new quality of energy, we also have to interpret $\left(\frac{\partial S_x}{\partial x}\right)$ as the momentum, p. This is just of course what is sometimes known as the "guidance condition", which Bohm (1953) himself took as a subsidiary condition. Notice then to construct our phase space we must abandon the insistence that only the eigenvalues of operators have physical meaning.

Now in the light of the results of section 3 we can see how the momentum relation arises naturally from the classical limit

$$S_x \to S_{cl}\ ,\ p \to p_{cl} = \left(\partial S_{cl}/\partial x\right),\ Q \to 0\,. \qquad (54)$$

Since we now have the possibility of a p-representation, we can ask what replaces the guidance condition? Clearly equation (53) shows that we must replace $-\left(\frac{\partial S_p}{\partial t}\right)$ with x. The resulting equation $x = -\left(\frac{\partial S_p}{\partial t}\right)$ clearly cannot be interpreted as a guidance condition and must be regarded as subsidiary conditions as originally suggested by Bohm himself (Bohm 1953). We can see that these conditions mean that we are only using the real part of the momentum and position respectively. In fact in the x-representation we have

$$\frac{e\left[\psi^*(x)P\psi(x)\right]}{|\psi(x)|^2} = \left(\frac{\partial S_x}{\partial x}\right) = p \tag{55}$$

while in the *p*-representation we have

$$\frac{e\left[\psi^*(p)X\psi(p)\right]}{|\psi(p)|^2} = -\left(\frac{\partial S_p}{\partial p}\right) = x \tag{56}$$

The fact that we use the real part in both these expression should not be too surprising since to obtain the Bohm equation (32) we had to take the *real* part of the Schrödinger equation under polar decomposition. Furthermore it is now clear why the quantum potential is not *ad hoc* but a necessary feature. It is necessary to ensure that both energy and momentum are conserved. For example the kinetic energy used in equation (52) is calculated from the *real* part of $\left(\psi^*(x,t)\hat{P}\psi(x,t)\right)^2 / 2m$ which is clearly not the quantum kinetic energy calculated from $\left(\psi^*(x,t)\hat{P}^2\psi(x,t)\right) / 2m$. In fact the difference is simply the quantum potential. Thus the conservation of energy is just

$$\frac{\partial S_x}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x^2 - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) = 0$$

In *p*-rep it is the potential energy, which is calculated from the *real* part of $\left(\phi^*(p,t)\hat{X}\phi(x,t)\right)^2$ cannot be the total potential energy which must be calculated from $\left(\phi^*(p,t)\hat{X}^2\phi(x,t)\right)$. Once again the difference is taken care of in the quantum potential Q_p. Thus the conservation of energy demands we write

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0$$

Thus I stress yet again, the quantum potential is a necessity and not *ad hoc*. Thus the criticisms that have been made by Heisenberg (1958) and Polkinghorne (2002) are based on a misunderstanding of the quantum formalism itself.

5.5 Shadow phase spaces.

Now we can see how the Bohm approach can be understood in terms of non-commutative geometry. If we give primary significance to the algebra of the operators then we cannot construct a phase space with the eigenvalues of the *X* and *P*-

operators. This is the position adopted in standard quantum mechanics. However we do not have to stop there. We can adopt the Gel'fand approach and project from the algebra onto shadow phase spaces. Thus in the two representations we have discussed we can construct two different but related phase spaces, both of which have trajectories as shown in figure 6. Of course these trajectories are just the streamlines of the corresponding probabilities as has been shown in detail in Brown and Hiley (2002). In that paper it is shown that the trajectories are nothing more or less than the probability currents. Thus the basic physical premise of the Bohm approach is simply to assume that if we retain the notion of a localised particle, then we can regard the particle as following the streamlines of the probability current, an assumption that is actually often made in physical problems in condensed matter physics, particularly in super-conductivity problems. (See, for example, Feynman *et al* 1965).

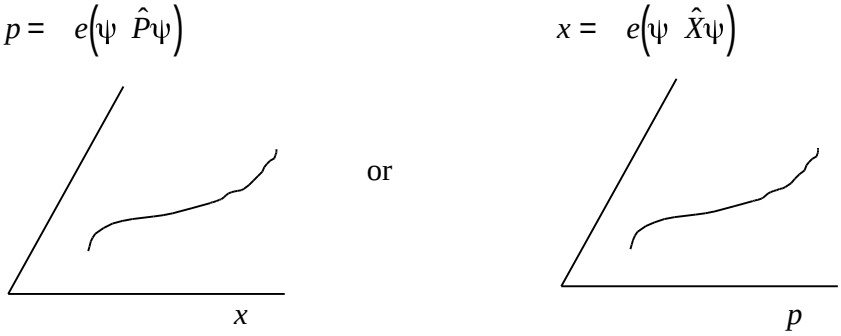

Figure 6

6. Conclusion.

In this paper I have tried to bring out the central role of the symplectic symmetry in both classical and quantum mechanics. Since the algebra of dynamical functions in classical mechanics is commutative there is no need to distinguish between dynamical operators and their eigenvalues. In quantum mechanics, on the other hand the algebra of dynamical operators, which carry the symplectic symmetry, is non-commutative. This means, in particular, that we cannot build an $x - p$ phase space out of the *eigenvalues* of these operators. Indeed these eigenvalues do not *directly* satisfy the symplectic symmetry, the symmetry of the eigenvalues are *enfolded* or *implicit* in the symplectic symmetry of the operators.

Similarity transformations of the type used in equations (37) and (38) show that each single position eigenvalue is 'exploded' into every momentum eigenvalue (and vice versa) under the transformation. This is the source of the probability in the quantum formalism. As long as we insist on identifying the physical properties with these eigenvalues we will always have this type of process occurring. It was this specific

mathematical structure that Bohm (1980) had in mind when he proposed the notion of the implicate order. In this view the non-commutative algebra *is* the implicate order. Each measurement makes manifest or explicates an eigenvalue that was enfolded in the algebra of operators. This is the origins of the notion of the implicate order. Thus each representation produces a specific explicate order. This is just another way of expressing what Pauli (1979) put in a more dramatic way.

> One can look at the world with the *p*-eye or with the *x*-eye, but if one wishes to open both eyes at the same time, one goes wrong.

It is not that 'one goes wrong', it is that the nature of quantum processes are such that it is not possible to manifest both aspects at the same time.

If we want to try to construct a phase space while the algebra of dynamical operators is taken to be primary then we must resort to constructing shadow manifolds. But this entails giving up the demand that the physical properties must always be characterised by their eigenvalues. We keep the idea that properties made manifest in a measurement are the eigenvalues of the appropriate operator, say, $\hat{A}$. But the complementary variable is not an eigenvalue of that complementary operator $\hat{B}$. Here we assume $\left[\hat{A},\hat{B}\right]\neq 0$. Instead of eigenvalues we use the real part of quantities of the type defined in expressions of the form $e\langle a|\hat{B}|a\rangle$. If these expressions are used in Schrödinger's equation then in order to ensure conservation of energy they must be accompanied by a supplementary energy. It is this energy that has traditionally been called a quantum potential. We see the quantum potential energy is an *internal* energy. Thus we can now see exactly why this potential energy is totally different from a classical potential and why it has no external source. (For detail properties of the quantum potential see Bohm and Hiley 1993).

In this way we see that the Bohm approach is deeply embedded in the quantum formalism and should not be emotionally dismissed out of hand as I have often come across. It should provoke no ideological battle. After all we cannot show empirically that a particle actually follows a trajectory; equally we cannot empirically demonstrate that it does not follow a trajectory. No experiment can decide between these two possibilities so that the question cannot be resolved by experiment. Surely the Bohm approach simply provides another perspective on quantum processes in general and should be used if it helps to clarify things. It does not usurp the standard approach, it merely shows that there is another way of looking at quantum phenomena which many find useful.

Acknowledgments.

I would like to thank Maurice de Gosson for a number of very stimulating discussions. I would also like to thank Melvin Brown for sharing with me much of his work that will be reported in his PhD thesis. I should also like to thank members of TPRU for critical debates during the development of these ideas.

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