

Phase Space Descriptions of Quantum Phenomena

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Abstract

In this paper we show the deep connection between the Wigner-Moyal approach and the Bohm approach to quantum mechanics. We point out that the key equations used in the Bohm approach were already contained in Moyal's classic 1949 paper. Furthermore we argue that these two approaches can be seen as different but related aspects of standard quantum formalism when the algebraic approach, rather than the Hilbert space approach is taken as primary. This leads us to consider the non-commutative Moyal *algebra in which two key equations obtained are shown to be the analogue of algebraic equations derived by Brown and Hiley using the algebraic approach. The different phase spaces exploited in each case appear as necessary consequences of the non-commutative algebra since they are but shadow manifolds already implicit in the algebra defining the quantum formalism.

1 Introduction.

As is well known that the non-commutativity of the Heisenberg algebra implies an uncertainty relationship $\Delta x \Delta p \simeq \hbar$, which is taken to mean that it is

not possible to represent a quantum process in a $x-p$ phase space. Nevertheless both the Wigner distribution theory [31] [29] [8] and the Bohm approach [1] [5] [24] use such phase space pictures. The Wigner approach proposes a phase space distribution function, $f(x, p, t)$ from which we can calculate the quantum probabilities exactly. The Bohm approach uses a different phase space in which the momentum of a particle is given by $p(x, t) = \nabla S(x, t)$. In this phase space particles are assumed to follow well-defined trajectories. The ensemble of trajectories that correspond to a given initial probability distribution produce a final probability distribution which is again exactly the same as that produced by the standard approach. Both are mathematically exact and neither add any new mathematical content.

In spite of all this there is an established feeling that these two approaches must be flawed in some way. Of course they provide different interpretations but the question that is central to this paper is how do they manage to construct phase spaces from the *same mathematics*? Notice I am deliberately putting the emphasis on mathematical structures. I know Wigner and Bohm give different interpretations, which may have difficulties with them, but for the purposes of this paper I am not interested in these interpretations. The arguments concerning interpretations get nowhere since they depend very much on personal preferences and prejudices. In this paper I want to examine how these approaches avoid the numerous ‘no go arguments *mathematically*’.

What I will show is that these two approaches are deeply imbedded in the standard quantum formalism. They are “not something completely different. They are part of the formalism. Furthermore there is a relationship between the two approaches. In fact I will show that the equations upon which Bohm built his interpretation are already contained in the classic paper of Moyal [29]. I will also show that the (x, p) used by Moyal have different meanings from those used by Bohm and these in turn are different from the eigenvalues of the operators \hat{x} and \hat{p} used by standard quantum mechanics. In particular we will show that the Bohm momentum is identical with the momentum that Moyal writes as \bar{p} , which is a mean momentum calculated from the Wigner distribution. We will see that the transport equation for derived by Moyal is just the real part of the Schrödinger equation used by Bohm. Thus the quantum potential already appears explicitly in the Moyal transport equation. Thus the Bohm approach is already part of the Wigner distribution method.

In the process of bringing all this out I have found it necessary to go on to consider the Moyal *product [10], which generates a deformed Poisson

algebra. The importance of this algebra is that in the limit of a small deformation parameter (in this case \hbar) this algebra becomes the Poisson algebra of classical mechanics. Indeed the star product can be written in terms of two brackets, the sine or Moyal bracket and the cosine or Baker bracket. In the limit the Moyal bracket becomes the Poisson bracket while the Baker bracket becomes the ordinary product.

While all this is well-known, what is not so well-known is that these two brackets have their exact analogue in the generalised algebraic formulation of quantum mechanics as outlined for example in Emch [14] or for the more mathematically inclined in Bratteli and Robinson [6]. I stumbled across the connection while attempting to solve a different puzzle. Melvin Brown and I [7] were trying to find where the quantum potential arose in the Heisenberg matrix form of quantum theory. We were led to a pair of algebraic equations, which were alternatives to the Schrödinger equation and its dual in the Hilbert space approach. One of these involved the commutator of the density operator and the Hamiltonian. This equation is the exact analogue of the equation of motion for the Moyal bracket.

The second equation involved the anticommutator or Jordan product of these objects. This equation is clearly related to the Baker bracket but the relation was not completely transparent. We will discuss this relation in more detail later in this paper. This second equation is particularly interesting because although it is an operator (representation-free) equation it contains no quantum potential. However once it is projected into a representation, a quantum potential immediately appears. Thus this potential is a consequence of projection into a Hilbert space and plays no *explicit* role in the algebraic structure as such. The projection splits the kinetic energy into two parts and the quantum potential is a result of this split. This suggests that it is a kind of internal energy as was discussed in Bohm and Hiley [5] and Hiley [17].

The appearance of these two equations enables us to support the idea that the algebraic formulation of quantum theory is a more appropriate way to proceed not only in quantum field theory but also in non-relativistic particle theory. This implies that the geometry we need for quantum mechanics is a non-commutative geometry where we do not start from an a priori given phase space. Instead we start with the non-commutative algebra itself and construct (or abstract) phase spaces from that structure in the spirit of Gel'fand [11].

Since we give primary relevance to a non-commutative algebra there is no unique underlying manifold so we must resort to constructing ‘shadow’

manifolds. What the Wigner and the Bohm approaches do is to construct different shadow manifolds. Indeed we show how to construct Bohm approaches for different representations, especially the p -representation thus showing that the Bohm approach does not lose the mathematical $x - p$ symmetry as Heisenberg [16] and Polkinghorne [30] maintain. Bohm simply chooses the x -representation because it is the representation in which all our measurements are preformed.

The structure of the paper is as follows. In section 2 we briefly recall the essential but simple mathematical structure upon which the Bohm interpretation is built. We then briefly review the Wigner-Moyal approach showing that the phase space used here is based on a cell structure in phase space, x being the mean position and p the mean momentum of the cell. In section 3 we show exactly how the Wigner and Bohm approaches are related. In section 4 we introduce the Moyal *product together with the Moyal and Baker brackets and show how classical mechanics emerges in the limit. In section 5 we re-derive the two algebraic equations of Brown and Hiley [7]. We show that these are equivalent to the *bracket equations discussed in section 4. We then show in detail how they are related. In the final section 6 we discuss the notion of shadow manifolds and its implications for quantum mechanics in general.

2 The Bohm approach

The general way in which the Bohm interpretation is presented appears extremely naive. One simply writes the wave function in polar form and then substitutes this expression into the Schrödinger equation. On separating out the real part of the resulting equation we find an equation that looks remarkably similar to the classical Hamilton-Jacobi equation

$$\frac{\partial S_c}{\partial t} + \frac{(\nabla S_c)^2}{2m} + V = 0 \quad (1)$$

where S_c is the classical action. In fact we find what we have previously called the quantum Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \left\{ \frac{(\nabla S)^2}{2m} - \frac{\nabla^2 R}{2mR} \right\} + V = 0 \quad (2)$$

Here S is the phase of the wave function and the additional term $Q = -(\nabla^2 R)/2mR$ has been traditionally termed the quantum potential.

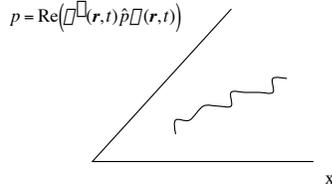


Figure 1: The Bohm phase space.

This additional term has generally been treated with suspicion and scepticism, being called *ad hoc* by Heisenberg [16] but we will see that it is absolutely essential if energy is to be conserved.

As is well known, trajectories exist in the classical limit as $S \rightarrow S_c$ and $Q \rightarrow 0$. As Q starts to increase there nothing to indicate at what value of Q the concept of a trajectory ceases to be applicable. In fact Hiley and Mufti [22] have shown that in the particular case of an inflationary universe, quantum trajectories flow smoothly into their classical counterparts as inflation increases to the limit. It therefore seems natural to assume that trajectories exist even at the quantum level. Indeed this is what Bohm [1] assumed and argued that these trajectories could be calculated from $p = \nabla S$ where S is the phase of the wave function. Notice that the trajectories are simply the streamlines of the probability flow.

Notice also that the p defined by Bohm is not an eigenvalue of the momentum operator, \hat{p} . It is in fact $\Re(\psi^*(\mathbf{r}, t)\hat{p}\psi(\mathbf{r}, t))$. Thus to conserve momentum we need the modified equation of motion

$$\frac{dp}{dt} + \nabla[V + Q] = 0 \quad (3)$$

which follows directly from equation (2). Equation (3) again contains an extra term, the quantum potential. Without this extra term momentum would not be conserved.

The imaginary part of Schrödinger's equation can be put into the form

$$\frac{\partial P}{\partial t} + \nabla \cdot (P \nabla S / m) = 0 \quad (4)$$

where $P = R^2 = |\psi|^2$ is the usual Born probability. This equation then ensures that probability is conserved. Thus if we start with a given initial

probability distribution both the Bohm approach and the standard approach give the same final probabilities. Thus as far as the mathematics is concerned both approaches produce identical results. More details of the Bohm approach can be found in Bohm and Hiley [3] [5] and in Holland [24].

3 The Moyal statistical theory

3.1 The Wigner distribution

Let us now move on to the Moyal approach. What he was trying to do was to find a probability distribution $f(X, P, t)$ in phase space so that the average of a quantum operator could be calculated from this distribution by averaging over all of phase space. Here we use capital X and P for reasons that will become clear shortly. Then the mean value of quantum operator $\hat{A}(\hat{x}, \hat{p}, t)$ takes the form

$$\langle \hat{A}(\hat{x}, \hat{p}, t) \rangle = \int A(X, P) f(X, P, t) dX dP \quad (5)$$

where the probability distribution is given by

$$f(X, P, t) = \frac{1}{2\pi} \int \psi^*(X - \eta\hbar/2) e^{-i\eta P} \psi(X + \eta\hbar/2) d\eta \quad (6)$$

This is the key equation of the Wigner approach and is known as the Wigner distribution function. The operator $\hat{A}(\hat{x}, \hat{p}, t)$ is replaced by the kernel of the integral

$$\hat{A}(\hat{x}, \hat{p}) = \frac{1}{(2\pi)^2} \int A(X, P) e^{i[\pi(\hat{x}-X)+\eta(\hat{p}-P)]} d\eta d\pi dX dP \quad (7)$$

This means that the mean value can be written in the form

$$\langle \hat{A} \rangle = \frac{1}{(2\pi)^2} \int \psi^*(x, t) A(X, P) e^{i[\pi(\hat{x}-X)+\eta(\hat{p}-P)]} \psi(x, t) d\eta d\pi dX dP dx \quad (8)$$

Once again this approach also produces *identical* results to the ones derived from the standard quantum formalism. The approach has always been seen as an attempt to provide a phase space distribution that would enable us to return to classical ideas. This attempt fails when we notice the distribution becomes negative and we are forced to talk about ‘quasi probability

distributions'. But the failure is a failure of an *interpretation* not of the mathematics as has been pointed out by Feynman [23]. All the mathematical moves used in arriving at equation (8) are exact. No approximations are used as we will show in more detail in section 3.4.

3.2 Relation between Bohm and Wigner

Now we want to show the exact relation between this approach and that of Bohm described in section 2. To do this we first outline the essential equations that are set out in detail in an appendix to Moyal's paper [29]. There he defines a mean powers of the momentum by

$$\begin{aligned} \rho(X)\bar{p}^n &= \int P^n f(X, P) dX \\ &= \left(\frac{1}{2i}\right)^n \left[\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^n \psi(x_1)\psi^*(x_2) \right]_{x_1=x_2=X} \end{aligned} \quad (A1.1)$$

(N. B. I will keep the same equation numbering system used by Moyal.) Thus for $\psi(x, t) = \sqrt{\rho} \exp[iS] = R \exp[iS]$ a straightforward calculation shows that the mean moment becomes

$$\bar{p}(x) = \frac{1}{2i} [\psi^* \nabla \psi - (\nabla \psi^*) \psi] = \nabla S \quad (A1.6)$$

Moyal then goes on to derive a set of what he calls transport equations. First he derives the transport equation for the probability density and eventually finds

$$\frac{\partial \rho}{\partial t} + \nabla \left(\rho \frac{\nabla S}{m} \right) = 0 \quad (A4.2)$$

This is just the conservation of probability equation (4) used by Bohm. Moyal then goes on to write down the transport equation for \bar{p} which he finds can be written in the form

$$\frac{\partial}{\partial t}(\rho\bar{p}) + \sum_i \frac{\partial}{\partial x_i} \left(\rho p_k \frac{\partial \bar{H}}{\partial x_1} \right) + \rho \frac{\partial \bar{H}}{\partial x_k} = 0 \quad (A4.3)$$

$$\frac{\partial}{\partial x_k} \left[\frac{\partial S}{\partial t} + \bar{H} - \frac{\nabla \rho}{8m\rho} \right] = 0 \quad (A4.4)$$

This means that

$$\frac{\partial S}{\partial t} + \overline{\overline{H}} = \frac{\nabla \rho}{8m\rho} \quad (\text{A4.5})$$

which can finally be written in the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V - \frac{\nabla^2 R}{2mR} = 0$$

This is identical to the Bohm equation (2) and is therefore the quantum Hamilton-Jacobi equation. Thus we see that the Bohm approach sits inside the Wigner-Moyal approach. The Bohm momentum is a mean momentum and corresponds to the current flowing along the streamlines. Bohm simply takes the Moyal approach one step further by assuming that individual particles follow these streamlines with momentum determined by the condition. It is now clear that the Bohm p is not the same as the Wigner-Moyal P . What then is P and what is X in the Wigner-Moyal approach? We will answer this question in the next section.

3.3 The Wigner distribution

We want to approach the Wigner distribution through the density matrix to show two things. (1) The Wigner distribution is simply an expression of the real part of the of the density matrix and therefore mathematically is a central part of the standard quantum formalism. (2) The Wigner distribution is constructed by first choosing a pair of operators (\hat{x}_1, \hat{p}_1) and (\hat{x}_2, \hat{p}_2) and then forming the mean operators $\hat{X} = (\hat{x}_2 + \hat{x}_1)/2$ and $\hat{P} = (\hat{p}_2 + \hat{p}_1)/2$. As we will show these operators commute and we can then form a phase space out of the simultaneous eigenvalues of these mean operators. To show this, consider a pure state where the density matrix can be written as $\rho(x_1, x_2, t) = \psi^*(x_1, t)\psi(x_2, t)$. I would like to emphasise that throughout this paper we only consider pure states even though we use the density matrix formalism. The generalisation to mixed states seems straightforward and will be considered in another paper.

Let us now write the wave function in the momentum representation using

$$\psi(x, t) = \frac{1}{\sqrt{(2\pi)}} \int \phi(p, t) e^{ipx} dp \quad (9)$$

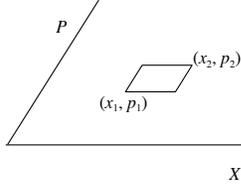


Figure 2: Cells in Wigner-Moyal phase space.

The density matrix can now be written in the form

$$\rho(x_1, x_2, t) = \frac{1}{2\pi} \int \int \phi^*(p_1, t) e^{-ix_1 p_1} \phi(p_2, t) e^{ix_2 p_2} dp_1 dp_2 \quad (10)$$

Now let us transform to the following coordinates

$$X = (x_2 + x_1)/2 \quad \eta = x_2 - x_1 \quad \text{and} \quad P = (p_2 + p_1)/2 \quad \pi = p_2 - p_1$$

then the transformed density matrix becomes

$$\rho(X, \eta, t) = \frac{1}{2\pi} \int \int \phi^*(P - \pi/2, t) e^{iX\pi} \phi(P + \pi/2, t) e^{i\eta P} dP d\pi \quad (11)$$

It is straightforward to show that $\rho(X, \eta, t)$ is real. We can now write equation (11) in the simplified form

$$\rho(X, \eta, t) = \int f(X, P, t) e^{i\eta P} dP \quad (12)$$

where

$$f(X, P, t) = \frac{1}{2\pi} \int \phi^*(P - \pi/2, t) e^{iX\pi} \phi(P + \pi/2, t) d\pi \quad (13)$$

This can then be put in the form of the Wigner distribution (6) by using the Fourier transformations

$$\phi^*(P - \pi/2, t) = \frac{1}{\sqrt{(2\pi)}} \int \psi^*(x_1, t) e^{i(P - \pi/2)x_1} dx_1$$

$$\phi(P + \pi/2, t) = \frac{1}{\sqrt{(2\pi)}} \int \psi(x_2, t) e^{-i(P + \pi/2)x_2} dx_2$$

Notice that this result is exact. We have simply transformed the density matrix in configuration space $\rho(x_1, x_2, t)$ into a density matrix in phase space

$f(X, P, t)$ built on the variables X and P . And again notice that we have made no approximations whatsoever.

Now we are in a position to explain what the coordinates X and P actually are. In fact X is an eigenvalue of the operator $\hat{X} = (\hat{x}_2 + \hat{x}_1)/2$, while P is an eigenvalue of the operator $\hat{P} = (\hat{p}_2 + \hat{p}_1)/2$. In this case they can be specified simultaneously because the operators commute, that is

$$[\hat{X}, \hat{p}] = \{[\hat{x}_2 + \hat{x}_1, \hat{p}_2 + \hat{p}_1]\} / 2 = 0$$

because $[\hat{x}_2, \hat{p}_2] = i$ while $[\hat{x}_1, \hat{p}_1] = -i$. These relations hold here because we have taken the two points in configuration space to be conjugate points. Thus we see the reason that the Wigner distribution method can construct a phase space is because it is using the mean position and mean momentum operators as its basic entities. The two points in configuration space allow us to do is to build cells in phase space. Equation (13) then shows that we are effectively averaging over these cells. Thus our distribution is averaged over regions in phase space and so a kind of non-locality is involved in this distribution. Bohm and Hiley [4] have already discussed some of the implications of this approach and have tried to construct positive probability distributions with some success. However that discussion was about interpretations. This present paper is about the mathematics underlying these approaches and we will continue to discuss these in the next section.

3.4 The central role of the Heisenberg group

It is important to realise what symmetry group lies behind the Wigner approach. This realisation will confirm that the mathematical structure is identical to the structure that lies behind the standard formalism. Let us recall that, apart from a phase factor, a general element of the Heisenberg group can be written in the form

$$\hat{G}(\eta, \pi) = \exp [i(\eta\hat{P} + \pi\hat{X})] \quad (14)$$

Then the expectation value of any element of this group is

$$g(\eta, \pi) = \langle \psi | \exp [i(\eta\hat{P} + \pi\hat{X})] | \psi \rangle \quad (15)$$

This then becomes

$$g(\eta, \pi) = \int \psi^* (X - \eta/2) e^{i\pi X} \psi (X - \eta/2) dX \quad (16)$$

which is simply the Fourier transform of the Wigner distribution given in equation (6). Thus the Wigner distribution is just the expectation value of a general element of the Heisenberg group and therefore we see the distribution is central to the standard quantum structure. It is also worth noting that, as Moyal [29] shows, the Wigner distribution can also be written as

$$f(X, P) = \frac{1}{\sqrt{\hbar}} \exp \left[-\frac{i}{2\hbar} \frac{\partial^2}{\partial P \partial X} \left[\psi^*(X) \psi(P) e^{iPX/\hbar} \right] \right] \quad (17)$$

(I have now ceased to assume $\hbar = 1$.) This in turn motivates the definition of the Moyal product

$$A * B = A(X, P) \exp \left[\frac{i\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial x} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overrightarrow{\partial}}{\partial x} \frac{\overleftarrow{\partial}}{\partial p} \right] \right] B(X, P) \quad (18)$$

which forms the basis of the Moyal or deformed Poisson algebra [10] [32] [13]. This algebra has some very important features that we will bring out in the next section.

4 Moyal algebra is the deformed Poisson algebra

I now want to move on to discuss briefly what I consider to be an essential algebraic structure that is central to the mathematical unification that I am considering in this paper. The algebra built on the Moyal $*$ product is one of a class of algebras known as deformed algebras known also as quantum groups [9] [27]. However for the purposes of this paper one does not need a detailed knowledge of this subject. From the $*$ product we can construct a Moyal (sine) bracket, which plays the role of the commutator in standard quantum mechanics. It can be written in the form

$$\{A, B\}_{MB} = \frac{A * B - B * A}{2i\hbar} = A(X, P) \sin \frac{\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial P} - \frac{\overrightarrow{\partial}}{\partial X} \frac{\overleftarrow{\partial}}{\partial P} \right] B(X, P) \quad (19)$$

Alongside the Moyal bracket, there is the Baker (cosine) bracket, which plays the role of the anti-commutator (or Jordan product) in standard quantum

mechanics. It is written in the form

$$\{A, B\}_{BB} = \frac{A * B + B * A}{2} = A(X, P) \cos \frac{\hbar}{2} \left[\frac{\overleftarrow{\partial}}{\partial X} \frac{\overrightarrow{\partial}}{\partial P} - \frac{\overrightarrow{\partial}}{\partial X} \frac{\overleftarrow{\partial}}{\partial P} \right] B(X, P) \quad (20)$$

For us here the importance of the Moyal bracket is that in the classical limit as $\hbar \rightarrow 0$ it becomes the Poisson bracket. That is

$$\{A, B\}_{MB} = \{A, B\}_{PB} + O(\hbar^2) \approx \left[\frac{\partial A}{\partial X} \frac{\partial B}{\partial P} - \frac{\partial A}{\partial P} \frac{\partial B}{\partial X} \right] \quad (21)$$

On the other hand the Baker bracket becomes the ordinary product

$$\{A, B\}_{BB} = AB + O(\hbar^2) \quad (22)$$

We can get a clearer idea of how these brackets fit into the picture if we first apply them to the simple case of stationary pure states. The pure state condition on the distribution f , like the condition on the density operator, is simply $f * f = f$. Following Curtright et al [10], let us consider the *-genvalues of the energy. We have the relations

$$H(x, p) * f(x, p) = Ef(x, p) \quad \text{and} \quad f(x, p) * H(x, p) = Ef(x, p) \quad (23)$$

By adding and subtracting the two equations in (23), we find immediately

$$\{H, f\}_{BB} = EF \quad \text{and} \quad \{A, B\}_{MB} = 0 \quad (24)$$

The second equation in (24) will immediately be recognised to have something to do with the Liouville equation for a stationary state. Indeed if we move on to consider a time dependent state then we find

$$\frac{\partial f}{\partial t} + \{A, B\}_{MB} = 0 \quad \implies \quad \frac{\partial f}{\partial t} + \{A, B\}_{PB} = 0 \quad (25)$$

This results suggests that the time dependent equation for the Moyal bracket is the analogue of the Liouville equation and expresses the conservation of probability. This leaves us with the question as to what equation the Baker bracket satisfies in non-stationary state. Fairlie and Manogue [15] have suggested that one needs two times in order to arrive at the appropriate equation. In the next section we will show that, in fact, we needn't introduce two times. A single time is sufficient. In fact in section 5.2 we will derive such an equation using the Baker bracket that reduces to the classical Hamilton-Jacobi equation in the limit of $\hbar \rightarrow 0$.

5 The generalised Heisenberg (algebraic) approach to quantum mechanics

5.1 The operator equations of motion

In this section we want to return to consider the algebraic approach to quantum mechanics and raise the following question: “What is the algebraic equivalent to the Schrödinger equation?” At first we thought that the answer lay with the well-known Heisenberg equation of motion. However we are not simply interested in making operators time dependent. The wave function still plays a roll even though it is frozen in time. This means we are not fully algebraic in the sense of Emch [14]. What we need is an algebraic element to play the role of the wave function. In other words we need a *wave operator*. This turns out to be just an element of the left ideal of the algebra.

The fully algebraic approach to quantum mechanics already uses the concept of a state that essentially involves the density operator, which in the Dirac notation is simply $\hat{\rho} = |\psi\rangle\langle\psi|$. But this still has vestiges of the Hilbert space structure and the aim of algebraic quantum mechanics is to be free initially of Hilbert space. One reason for this generalisation occurs in quantum field theory where as soon as an interaction is switched on the system is transformed from the free particle Hilbert space to an inequivalent Hilbert space. Such a transformation cannot be handled by the ordinary methods. Although this generalisation is generally regarded as only relevant for field theory I was curious to see what would result if we applied the same techniques to the one particle problem. In pursuing such a question Brown and I [7] discovered a new algebraic (operator) equation, which turns out to be the algebraic generalisation of the equation involving the Baker bracket as I will now show.

The density operator is essentially the operator equivalent of the amplitude of the wave function. But where is the operator equivalent of the phase of the wave function? In order to get a handle on this we use a clue provided by Dirac [12]. In the third edition of his book he introduces what he calls the standard ket. This is not the ket, $|\ \rangle$, that is used throughout physics. It is denoted by the symbol \rangle and has been barely noticed. It was, in fact, Dirac’s attempt to introduce a primitive idempotent into the Heisenberg algebra.

To show how it works let us write

$$\hat{\rho} = \hat{\psi}\rangle\langle\hat{\phi} = \hat{\psi}\epsilon\hat{\phi} = \hat{\psi}_L\hat{\phi}_R = \hat{A}\epsilon\hat{B} \quad (26)$$

Here $\hat{\psi}$ and $\hat{\phi}$ are wave operators, ϵ is an idempotent so that $\epsilon^2 = \epsilon$. $\hat{\psi}_L = \hat{A}\epsilon$ ($\hat{\phi}_R = \epsilon\hat{B}$) is a left (right) ideal in the algebra. The left ideal is the ‘operator’ form of the wave function, while the right ideal is the ‘operator’ form of the conjugate wave function. These are just the symplectic analogues of the Pauli or Dirac spinors of the orthogonal Clifford algebra. They are symplectic spinors. More details of these algebraic objects will be found in Hiley [19] [20]. A very interesting use of these idempotents will be found in Kauffman [25] [26]. For the purposes of this paper we need start with two algebraic (operator) Schrödinger-type equations.

$$i\frac{\partial\hat{\psi}_L}{\partial t} = \hat{H}\hat{\psi}_L \quad \text{and} \quad -i\frac{\partial\hat{\psi}_R}{\partial t} = \hat{\psi}_R\hat{H} \quad (27)$$

The motivations for this choice will be found in Hiley [19] [21]. We then proceed as in section 4 and sum these two equations to obtain

$$i\left[\left(\frac{\overrightarrow{\partial}}{\partial t}\hat{\psi}_L\right)\hat{\psi}_R + \hat{\psi}_L\left(\hat{\psi}_R\frac{\overleftarrow{\partial}}{\partial t}\right)\right] = \left(\overrightarrow{\hat{H}}\hat{\psi}_L\right)\hat{\psi}_R - \hat{\psi}_L\left(\hat{\psi}_R\overleftarrow{\hat{H}}\right) \quad (28)$$

By writing $\hat{\rho} = \hat{\psi}_L\hat{\psi}_R$ we obtain the equation

$$i\frac{\partial\hat{\rho}}{\partial t} + [\hat{\rho}, \hat{H}]_- = 0 \quad (29)$$

which we immediately recognise as the Liouville equation expressing the conservation of probability. Clearly this is the operator analogue of the equation (25) where the Moyal bracket has been replaced by the commutator. If we now subtract the two Schrödinger-like equations we find

$$i\left[\left(\frac{\overrightarrow{\partial}}{\partial t}\hat{\psi}_L\right)\hat{\psi}_R - \hat{\psi}_L\left(\hat{\psi}_R\frac{\overleftarrow{\partial}}{\partial t}\right)\right] = \left(\overrightarrow{\hat{H}}\hat{\psi}_L\right)\hat{\psi}_R + \hat{\psi}_L\left(\hat{\psi}_R\overleftarrow{\hat{H}}\right) \quad (30)$$

Now let us assume that the left and right ideals can be polar decomposed in the form

$$\hat{\psi}_L = \hat{R}e^{i\hat{S}}\epsilon \quad \text{and} \quad \hat{\psi}_R = \epsilon e^{i\hat{S}}\hat{R} \quad (31)$$

The equation can easily be shown to reduce to the form

$$\hat{\rho}\frac{\partial\hat{S}}{\partial t} + \frac{1}{2}[\hat{\rho}, \hat{H}]_+ = 0 \quad (32)$$

This is just a phase equation, which as we will see below provides an analogue of the time-dependent equation involving the Baker bracket. We will also see that this equation is actually an expression of the conservation of energy equation. Equations (29) and (32) were first obtained in Brown and Hiley [7]. There we showed that equation (32) presents no problem for gauge invariance and in fact we easily derived the Aharonov-Bohm effect and the Berry phase directly from it. This was all spelt out in detail in Brown and Hiley [7]. We also showed there that although the quantum potential is not present in the algebraic form of equation (32), the quantum potential immediately appears when we project the equation into a specific representation. We will discuss this in section 6.

5.2 Relation to the Wigner distribution approach

We have already remarked at the possibility of constructing a time dependent equation involving the Baker bracket and thus establishing a deeper relationship between algebraic quantum mechanics and the Wigner-Moyal approach. To this end let us consider the following two non-stationary (pure) state equations.

$$H * f = \frac{i}{2\pi} \int e^{-inp} \psi^*(x - \hbar\eta/2) \frac{\partial \psi(x + \hbar\eta/2)}{\partial t} d\eta \quad (33)$$

$$f * H = -\frac{i}{2\pi} \int e^{-inp} \frac{\partial \psi^*(x - \hbar\eta/2)}{\partial t} \psi(x + \hbar\eta/2) d\eta \quad (34)$$

These are clearly the direct analogue of the two Schrödinger-type equations (27). If we take the difference of equations (33) and (34) we immediately obtain the Liouville equation (25). If on the other hand we take the sum we get

$$H * f + f * H = 2\{H, f\}_{BB} = \frac{i}{2\pi} \int d\eta e^{-inp} \left[\psi^*(x - \hbar\eta/2) \frac{\partial \psi(x + \hbar\eta/2)}{\partial t} - \frac{\partial \psi^*(x - \hbar\eta/2)}{\partial t} \psi(x + \hbar\eta/2) \right] \quad (35)$$

Again we can simplify this equation by writing $\psi = Re^{iS}$ when we find the integrand becomes

$$[\psi^* \partial_t \psi - (\partial_t \psi^*) \psi] = \left[\frac{\partial_t R(x + \hbar\eta/2)}{R(x + \hbar\eta/2)} - \frac{\partial_t R(x - \hbar\eta/2)}{R(x - \hbar\eta/2)} \right]$$

$$+i \left[\frac{\partial_t S(x + \hbar\eta/2)}{S(x + \hbar\eta/2)} + \frac{\partial_t S(x - \hbar\eta/2)}{S(x - \hbar\eta/2)} \right] \psi^* \psi \quad (36)$$

Remarkably (and satisfyingly) after some straightforward manipulation we find that in the classical limit equation (35) becomes simply

$$H * f + f * H = 2\{H, f\}_{BB} = -2\frac{\partial S}{\partial t} f + O(\hbar^2) \quad (37)$$

which will immediately be recognised as the classical Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0 \quad (38)$$

because of equation (22) viz. $\{A, B\}_{BB} = AB + O(\hbar^2)$. Thus we see that that the equation (35) is simply the Wigner-Moyal equivalent of equation (32)

$$\hat{\rho} \frac{\partial \hat{S}}{\partial t} + \frac{1}{2} [\hat{\rho}, \hat{H}]_+ = 0 \quad (32)$$

Now as we will now show equation (32) will produce the quantum Hamilton-Jacobi equation like equation (2) thus reconfirming that the Bohm formalism is at the centre of standard quantum mechanics.

6 Non-commutative geometry and shadow manifolds

Before moving on to consider the more general implications of the algebraic approach we want to produce the quantum Hamilton-Jacobi equation from equation (32). Let us first project into a general representation so that equation (29) becomes

$$i \frac{dP(a)}{dt} + \langle [\hat{\rho}, \hat{H}]_- \rangle_a = 0 \quad (39)$$

while equation (32) becomes

$$P(a) \frac{\partial S(a)}{\partial t} + \frac{1}{2} \langle [\hat{\rho}, \hat{H}]_+ \rangle_a = 0 \quad (40)$$

To see how these equations lead to equations (2) and (4) of the Bohm approach let us choose the x -representation and choose for convenience the harmonic oscillator Hamiltonian.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2} \quad (41)$$

Then equation (39) becomes equation (4)

$$\frac{\partial P}{\partial t} + \frac{\partial(Pv)}{\partial x} = 0 \quad (4)$$

However equation (40) becomes

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m} \left(\frac{\partial S_x}{\partial x} \right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x} \left(\frac{\partial^2 R_x}{\partial x^2} \right) = 0 \quad (42)$$

which is just the harmonic oscillator version of equation (2). Here R_x and S_x are the amplitude and phase of the wave function in the x -representation. Thus the quantum potential $Q = \frac{1}{2mR_x} \left(\frac{\partial^2 R_x}{\partial x^2} \right)$ appears only when the algebraic equation is projected into a specific representation. If we identify $p = (\partial S_x / \partial x)$ we see that this equation is again simply an expression for the conservation of energy. The fact that the quantum potential only appears on choosing a representation can be confirmed by going to the momentum representation. In this case equations (29) becomes

$$\frac{\partial P_p}{\partial t} + \frac{\partial j_p}{\partial p} = 0 \quad (43)$$

while equation (32) becomes

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2} \left(\frac{\partial S_p}{\partial p} \right)^2 - \frac{K}{2R_p} \left(\frac{\partial^2 R_p}{\partial p^2} \right) = 0 \quad (44)$$

Here R_p and S_p are the amplitude and phase of the wave function in the p -representation. If we identify $x = -(\partial S_p / \partial p)$ we then see that a quantum potential $Q = -\frac{K}{2R_p} \left(\frac{\partial^2 R_p}{\partial p^2} \right)$ appears once again as a consequence of the projection. This approach restores the full mathematical $x - p$ symmetry to the Bohm approach thus removing another objection raised by Heisenberg [16] in his original criticisms of the Bohm approach. The choice of the x -representation as being the basic representation is then to be a physical

assumption and is not necessary in general. Indeed such a position fits much more naturally into Bohm's more general notion of the implicate order [2].

We will not discuss this aspect further in this paper. We will finally remark that the trajectories are now simply the streamlines of the respective probability currents given in the x -representation by

$$j_x = \langle x | \frac{\partial(\rho p^2/2m)}{\partial p} | x \rangle = \Re[\psi^*(x)\hat{p}\psi(x)] / |\psi(x)|^2 = \frac{\partial S_x}{\partial x} \quad (45)$$

and in the p -representation by

$$j_p = -\langle p | \frac{\partial(\hat{\rho}V(\hat{x}))}{\partial x} | p \rangle \quad (46)$$

The currents are clearly more complicated to evaluate in the p -representation, but we will not go further into this question here. A more detailed discussion of these currents will be found in Brown and Hiley [7].

7 Conclusion

What we have tried to bring out throughout this paper is the importance of algebraic structures in the quantum formalism. In particular we see that there are two types algebra both non-commutative that are deeply related to each other. One is the standard algebraic structure used in more generally in quantum field theory. This puts emphasis on the algebraic structure rather than Hilbert space structures as do the more conventional approaches. The other is the Moyal $*$ algebra, which is based not on operators but on functions. Nevertheless it is still non-commutative. Both of these algebras give rise to a more general structure that enables quantum mechanics to directly encompass thermal properties, which cannot be handled by a single Hilbert space formalism. However in this paper we have seen further advantages of using this generalisation even for the case of single particles in pure states in the non-relativistic case.

Indeed we have seen in this paper that there are remarkable similarities between the Wigner-Moyal approach based on the $*$ product algebra and the operator algebra when the wave functions are replaced by the more general concept of left and right ideals of the algebra. We have shown all of this in the case of pure states and will leave the discussion to mixed states to another paper.

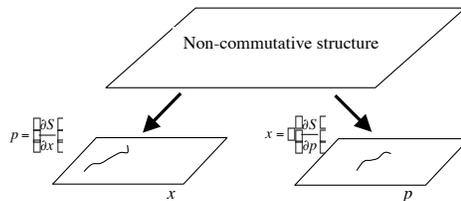


Figure 3: Projections onto shadow manifolds.

For now we just want to make some remarks about the way we see the algebraic approach fitting into a general physical picture of quantum phenomena. Monk and Hiley [28] have already sketched a possible scenario and we will build on that here. We want to take the algebra as primary and regard the representation as secondary. In physical terms this means that we do not start with the classical phase space and then try to build a structure on a particular *a priori* given phase space. Rather we want the algebraic structure, which comprise the observables to determine what possible underlying spaces this algebra supports.

For a commutative algebra this question has been answered by Gelfand (See Demaret *et al* [11] for a simple account of what is actually involved.) What Gelfand showed was that you could either start with an *a priori* given manifold and construct a commutative algebra of functions upon it or one could start with a given commutative algebra and deduce the properties of a unique underlying manifold. If the algebra is non-commutative it is no longer possible to find a unique underlying manifold. The physicist's equivalent of this is the uncertainty principle when the eigenvalues of operators are regarded as the only relevant physical variables.

What the mathematics of non-commutative geometry tells us is that in the case of a non-commutative algebra all we can do is to find a collection of shadow manifolds. The phase spaces constructed by the methods of Wigner and Bohm are then examples of these shadow manifolds. Thus we should not regard Wigner and Bohm as lying outside the quantum formalism. They are central to it. They are simply constructing different representations of the same formalism. The emotional arguments that surround discussions about the Wigner and Bohm approaches are totally misplaced. The appearance of shadow manifolds is a necessary consequence of the non-commutative structure of the quantum formalism.

In philosophical terms this is just what David Bohm was getting at in his notion of the implicate order[2]. The order of process by its very essence cannot be displayed in one unique manifest (explicate) order. The nature of quantum processes is such that its very essence is such that we can only display some aspects of the process at the expense of others. We are inside looking out. We are participating in Nature. We are participating through our instruments. Because of our activity not everything can be made manifest together. The interminable arguments about the interpretation of quantum mechanics stem from an attempt to find a view from outside. It is not that the quantum formalism is underdetermined. All we can have is the process described as an implicate order through its algebraic structures. All we see is one physical process projected into another. There is no unique ‘God’s’ eye view of the Universe except at the classical level.

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References

- [1] Bohm, D., Phys. Rev., **85**, 166-179; **85**,180-193 (1952).
- [2] Bohm, D., *Wholeness and the Implicate Order*, Routledge, London. (1980).
- [3] Bohm, D and Hiley, B. J., Phys. Reports **144**, 323-348, (1987).
- [4] Bohm, D. and Hiley, B. J., Found. Phys., **11**, 179-203, (1981).
- [5] Bohm, D. and Hiley, B. J., *The Undivided Universe : an Ontological Interpretation of Quantum Theory*, Routledge, London, (1993).
- [6] Bratteli, O. and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics I*, Springer, Berlin, (1979).

- [7] Brown, M. R. and Hiley, B. J., quant-ph/0005026, (2000).
- [8] Carruthers, P. and Zachariasen, F., Rev. Mod.Phys., **55**, 245-85, (1983).
- [9] Chari, V. and Pressley, A., *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, (1995).
- [10] Curtright, T., Fairlie, D., and Zachos, C., Phys. Rev. **D58**, 025002-1-14, (1998).
- [11] Demaret, J., Heller, M. and Lambert, D. , Foundations of Science, **2**, 137-176, (1997).
- [12] Dirac, P. A. M., *The Principles of Quantum Mechanics*, Oxford University Press, Oxford, (1947).
- [13] Dubin, D., Hennings, M. A. and Smith, T. B., *Mathematical Aspects of Weyl Quantization and Phase*, World Scientific, Singapore, (2000).
- [14] Emch, G. G., *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley-Interscience, New York, (1972).
- [15] Fairlie, D. and Manogue, C., J. Phys **A24**, 3807-3815, (1991).
- [16] Heisenberg, W., 1958 *Physics and Philosophy : the revolution in modern science*, George Allen and Unwin , London, (1958).
- [17] Hiley, B. J., *Computing Anticipatory Systems : CASYS 2000-Fourth International Conference*, edited by D. M. Dubois, pp. 77-88, AIP, (2001).
- [18] Hiley, B. J., Conference Proceedings *Quantum Theory : Reconsideration of Foundations*, ed. A. Khrennikov, pp. 141-162, Växjö University Press, Sweden, (2002).
- [19] Hiley, B. J., *Correlations*, ANPA**23**, ed K. G. Bowden, 104-134, (2002).
- [20] Hiley, B. J., *Boundaries , Scientific Aspects of ANPA24*, ed K G Bowden, pp 149-186 (2003).
- [21] Hiley, B. J., *Non-commutative Quantum Geometry : a reappraisal of the Bohm approach to quantum theory*, to appear in Quo Vadis Quantum Mechanics, (2004).

- [22] Hiley, B. J. and Aziz Mufti, A. H., *Fundamental Problems in Quantum Physics*, ed., M. Ferrero and A. van der Merwe, pp.141-156, Kluwer, Dordrecht, (1995).
- [23] Hiley, B. J. and Peat, F. D., *Quantum Implications:Essays in Honour of David Bohm*, Routledge, London, (1987).
- [24] Holland, P., *The Quantum Theory of Motion*, Cambridge University Press, Cambridge, (1993).
- [25] Kauffman, L. H., *Biologic*, AMS Contemporary Mathematics Series, **304**, 313 - 340, (2002).
- [26] Kauffman, L. H., *Quantum Computing and the Jones Polynomial*, AMS Contemp. Math. Series, **305**, 101-137, (2002).
- [27] Majid, S., *A Quantum Group Primer*, Cambridge University Press, Cambridge, (2002).
- [28] Monk, N. and Hiley, B. J., *Found. Phys. Letts*, **11**, 371-377,(1998).
- [29] Moyal, J. E., *Proc. Camb. Phil. Soc.*, **45**, 99-123, (1949).
- [30] Polkinghorne, J. *Quantum Theory : a very short introduction*, Oxford, (2002).
- [31] Wigner, E. P., *Phys. Rev.*, **40**, 749-59, (1932).
- [32] Zachos, C., hep-th/0110114, (2002).