Generalisation of the twistor to Clifford algebras as a basis for geometry *

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Abstract

In this paper, we generalize the Penrose twistor theory to a Clifford algebra. This allows basic geometric forms and relationships to be expressed purely algebraically. In addition, by means of an inner automorphism of this algebra, it is possible to regard these forms and relationships as emerging from a deeper pre-space, which we are calling an implicate order. The way is then opened up for a new mode of description, that does not start from continuous space-time, but which allows this to emerge as a limiting case.

1 Introduction

Because of the complex and insoluble problems that arise in the attempt consistently to formulate the basic concepts in quantized general relativity, Wheeler [1] and t'Hooft [2] have questioned whether continuous space-time is the right starting point. In particular, Wheeler has suggested a notion of pre-space, based on something deeper than geometry, from which geometry is to emerge as a suitable limiting case. One of us [3,4] (D.B.) has proposed the related notion of the enfolded or implicate order, as a way of understanding the meaning of the quantum

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theory. Since the difficulties in general relativity arise in the attempt to quantize this theory, it seems reasonable to suggest that pre-space is a form of the implicate order. (Rather than, for example, a simple “foam-like structure, as Wheeler and Hawking [5] have assumed.) The implicate order may thus arise as a natural point of departure for discussing what is meant by pre-space.

Now, any theory based on an algebra can always be put in an implicate order by an inner automorphism of the algebra [3,4] . If A is an algebraic element, and T is the automorphism, then $A' = TAT^{-1}$ which is, however, enfolded in an implicate order. It is well known that relativistic quantum mechanics can be expressedcompletelythroughtheinterweavingofthreebasicalgebras,thebosonic, the fermionic and the Clifford. This means that the whole of relativis ic quantum mechanics can also be put into an implicate order, as has indeed already been suggested in previous references [3,4].

Quantum mechanics has thus far generally been regarded solely as a theory of physical entities, such as particles and fields, so that its implications for geometry can, at most, come out only indirectly through the quantization of the gravitational potentials. In such quantization, the continuous (unfolded) order of space-time is taken for granted (as expressed in terms of a co ordinate system). The notion of pre-space requires, however, that we start from more fundamental considerations, inthe sensethatthe structureofquantum relationshipsappliesdirectly tothedeeper level underlying geometry. Out of these quantum relationships in pre-space must then emerge the familiar geometric forms and structures, such as points, lines, planes, etc., with their magnitudes and orientations, but only as statistical averages in some limiting case. Or to put it differently, these forms and structures will unfold from the implicate order. More generally, from the very beginning, we will be dealingbasically with animplicate order, from whichthe explicate (or unfolded) order of ordinary space and time will come out.

To achieve this goal mathematically, we propose to begin with our interwoven bosonic, fermionic, and Clifford algebras as the basic mode of describing all geometric entities. These will generally be in an implicate order, and thus “spread out over a region of space and time determined by a suitable inner automorphism of the algebra. However, a special case of this will be the explicate (or extended) order of ordinary geometrical forms, that are distinctly separated from each other.

This is, of course, a vast programme. Here,we shall make only a beginning, by generalizing and extending Penroses notion of a twistor. Such a twistor mathematically represents both a null ray and its location (or moment) relative to the origin. This can be displaced by changing its moment. The theory evidently allows for the entire Poincare group (and indeed even the conformal group). In it, a point is defined as the intersection of two null rays, which is expressed by means of a simple condition (given by eqn. (9)).

Our first step is to express the twistor in terms of the Clifford algebra (so that it can be transformed into an implicate order). This expression, however, generalizes the meaning of a twistor in several important ways, which will come out as we go along. In particular, we will be able to describe certain geometric structures, going beyond what can be done in twistor theory (e.g., we can extend the theory to include non-null rays, and various geometric forms, such as triangles, tetrahedra, etc.). Going on from here, we can now take account of trajectories, which are irregular curves resembling Brownian motion. These are shown to have a certain "twist" or helicity, corresponding to a kind of intrinsic angular momentum. On the other hand, in the Penrose approach, the nearest geometrical analogy to helicity is given only by an indirect reflection in a family of null rays associated with the null ray inquestion (calledaRobinsoncongruence). There isno directrepresentation of helicity (though there is a formal spin angular momentum associated dynamically to particles, rather than to trajectories).

In the work that follows, we show that a wide range of geometric forms and relationships can be described in terms of the Clifford algebra. This range is so wide that it seems reasonable to suppose that all basic geometrical forms and relationships can be put in this way, and therefore can be regarded as unfolding from an implicate order of pre-space. Because the property of chirality is included in this, the possibility is opened up for an explanation of parity non-conservation in β-decay as grounded in the properties of pre-space.

Of course, this approach requires much more development before it is ready to consider definite physical problems. We are, for the moment, however involved merely in developing a new description of the "arena" in which physical processes take place. When this is all fairly clear, then we can lookat actual physical processes as variations within the pre-space away from what can be called the normal background, or "vacuum state". However, before doing this, we have evidently to incorporate both relativity and quantum field theories along with general relativity as limiting cases of structures and relationships in pre-space. But in this paper, we shall begin by reviewing Penrose's work and go on to generalizing it in the ways described above.

2 Summary of twistors

According to the definition given by Penrose [6], a twistor is constituted of a pair of spinors (ω^A, π^A) that are in a certain relationship. The relationship is

$$\omega^A = ix^{A\dot{A}}\pi_{\dot{A}} \tag{1}$$

ln the usual notation for the Pauli spinor

$$x^{A\dot{A}} \leftrightarrow (x^0 - x.\sigma)^{A\dot{A}} \tag{2}$$

where x^0, x is an arbitrary four-vector.

The interpretation given by Penrose to the twistor is that it determines a null ray. To demonstrate this, let us first consider the special case for which $\omega^A = o$ Clearly, this implies that $x^{A\dot{A}} = 0$, or that $x^0 = 0, x = 0$, which represents the origin. But $\omega^A = 0$ will also be satisfied for any $x^{A\dot{A}}$ such that $x^{A\dot{A}}\pi_A = 0$ or $(x^0 - x.\sigma)^{A\dot{A}} = 0$. Evidently, this defines a null vector, (V^0, V) given in spinor form by $\bar{\pi}^A\pi^{\dot{A}}$.

Let us now consider a non-zero value of ω^A given by $\omega^A = iy^{A\dot{A}}\pi_{\dot{A}}$ so we have

$$\omega^A = ix^{A\dot{A}}\pi_A = iy^{A\dot{A}}\pi_{\dot{A}}$$

or

$$(x^{A\dot{A}} - y^{A\dot{A}})\pi_{\dot{A}} = 0$$

$x^{A\dot{A}} - y^{A\dot{A}}$ will now correspond to a null ray whose direction is given by $\bar{\pi}^A\pi^{\dot{A}}$, which passes through the point corresponding to $y^{A\dot{A}}$, it follows that a specification of the twistor $(\omega^A, \pi_{\dot{A}})$ determines the general null ray.

Penrose then introduces the twistor invariant. Writing $Z^\alpha = (\omega^A, \pi_{\dot{A}})$, we obtain for the invariant

$$J = Z^\alpha \bar{Z}_\alpha \tag{3}$$

Or

$$J = \omega^A\bar{\pi}_A + \bar{\omega}^{\dot{A}}\pi_{\dot{A}} \tag{4}$$

Z^α contains four complex numbers, with eight real parameters.

However, all geometrically interpreted quantities in the twistor theory are invariant to multiplication of Z^α by a complex factor. This leaves only six geometrically significant factors.

It follows from the definition (1) of a twistor that

$$J = i(x^{A\dot{A}}\pi_{\dot{A}}\bar{\pi}_A - x^{A\dot{A}}\bar{\pi}_A\pi_{\dot{A}}) = 0 \tag{5}$$

Sothereareonly fiveindependentparameters and this isjust thenumber needed to deflne a null ray (e.g., the two-directional parameters, and the three parameters representing its intersection with the hyperplane, $t = 0$). The more general spinor pair $(\omega^A, \pi_{\dot{A}})$ not satisfying $J = 0$ does not represent a null ray. We shall discuss the interpretation that Penrose gave to this later.

If we take two twistors, Z_1^α and Z_2^α the twistor scalar product is

$$J_{12} = Z_1^\alpha \bar{Z}_{2\alpha} = \omega^A{}_1\bar{\pi}_{2A} + \bar{\omega}_2^{\dot{A}}\pi_{1\dot{A}} \tag{6}$$

which is shown by Penrose to be conformally invariant. In the spinor notation, this becomes

$$J_{12} = ix_1^{A\dot{A}}\pi_{1\dot{A}}\pi_{2A} - \bar{x}_2^{A\dot{A}}\bar{\pi}_{2A}\pi_{1\dot{A}} \tag{7}$$

But because $x^{A\dot{A}}$ is a hermitean matrix, $\bar{x}_2^{A\dot{A}}\bar{\pi}_{2A}\pi_{1\dot{A}} = x_1^{A\dot{A}}\pi_{1\dot{A}}\pi_{2A}$ Therefore we obtain

$$J_{12} = i\pi_{1\dot{A}}\bar{\pi}_{2A}(x_1^{A\dot{A}} - x_2^{A\dot{A}}) \tag{8}$$

Evidently, for $x_1^{A\dot{A}} = x_2^{A\dot{A}}$, $J_{12} = 0$ The condition that two twistors correspond to intersecting light rays is thus

$$J_{12} = 0 \tag{9}$$

If we are seeking to start with extended strucures as our basic concept, and to regard the point as an abstraction from these, the twistor seems to afford a good point of departure for the development of a mathematical basis for such an approach It is clear, for example, that space-time displacements (as well as, of course, Lorentz transformations and even conformal transformations) can now be expressed entirely in terms of spinor quantities. Thus, a displacement $(\Delta x_1^0 \Delta x$ corresponding to $\Delta x^{A\dot{A}}$ leads to the formula

$$\Delta\omega^A = i(\Delta x)^{A\dot{A}}\pi_{\dot{A}} \tag{10}$$

Or, for a general function $\psi(\omega^A \pi_{\dot{A}})$ a space displacement is given by

$$\Delta\psi = \Delta x^{A\dot{A}}\pi_{\dot{A}}\frac{\partial\psi}{\partial\omega^A} \tag{11}$$

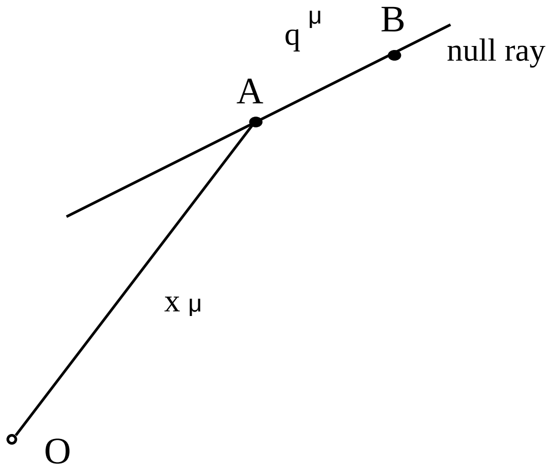

Figure 1:

Such displacement operations applied, for example, to a pair of twistors satisfying $J_{12} = 0$ describing intersecting null rays, will correspondto anarbitrary translation of both ofthem, incudingtheirpointof intersection, alongwiththe wholenullcone, defined by this point.

For Z^{α} not satisfying $J = 0$, Penrose has proposed a further interpretation. We let Y^{α} represent a twistor satisfying the twistor condition $Y^{\alpha}\bar{Y}_{\alpha}$. We write $J_{Z,Y} = 0$ for a fixed Z^{α} this will define a family of rays, which is called a Robinson congruence. This family may be seen to have a relationship between rays that impies a twist (as a twist is implied in the family of lines that make up a ruled hyperbolic surface). However, this interpretation brings in the twist only indirectly through its reflection in the family of rays making up the Robinson congruence. Bue the rays in the family are not themselves interpreted as part of any concrete geometric structure. The geometric meaning of this family of null rays is thus not clear. And so, the attractive simplicity of the original idea of having spinors directly represent geometric objects has been lost.

3 Twistors through Clifford Algebra

Let us consider a null ray, and in the way suggested by Penrose, connect this with the origin, through an arbitrary four vector, x^{μ}. Wenow add to Penroses suggestion and insert within the null ray a definite null vector, q^{μ}, which is supported at the end of the vector, x^{μ}, as shown in Fig. 1. The form can be represented through the Clifford algebra starting from its generators, such as (q^{μ}, γ_{μ}), To do this, we introduce the matrix τ defined by

$$\tau = (x^\mu \gamma_\mu)(q^\nu \gamma_\nu) = (x^\mu q_\mu)1 + \frac{1}{2}(x^\mu q^\nu - x^\nu q^\mu)\gamma_\mu \gamma_\nu \tag{12}$$

The matrix τ determines the moment

$$M^{\mu\nu} = x^\mu q^\nu - x^\nu q^\mu \tag{13}$$

of the vector q^μ around the origin, along with the scalar product

$$S = (x^\mu q_\mu) \tag{14}$$

We now define a **matrix pair** (τ, q) with $q = (q^\mu \gamma_\mu)$

The matrix τ determines the moment, $M^{\mu\nu}$, of the null vector, q^μ. From the space components of this moment, and from q^μ itself, we can determine the projection of the corresponding null ray in the hyperplane, $t = 0$. (See Fig. 2) For convenience we have taken the plane of AB to be that of the paper. The moment is then perpendicular to this plane, and is equal in magnitude to

$$M = |q||x|sin\theta$$

Clearly if M and $|q|$ are known, the distance, $d = |x|sin\theta$ from O to the line AB is known.

To determine the null ray completely, we need also to know its intercept with the hyperplane, t = 0. This is given in terms of the time components,

$$M^{\mu 0} = x^\mu q^0 - q^\mu x^0 = q^0(x^\mu - \frac{q^\mu}{q^0}x^0)$$

Evidently, the intercept is just

$$\xi^{\mu 0} = \frac{M^{\mu 0}}{q^0} = (x^\mu - \frac{q^\mu}{q^0}x^0) \tag{15}$$

This information is, of course, not enough to fix the point of support, x^μ, of the null vector, q^μ . For if we add $\Delta x^\mu = \alpha q^\mu$ to x^μ, we do not change the moment, nor do we change the scalar product since what is added is just $\alpha(q^\mu q_\mu) = 0$. So our matrix pair leaves the point of support of x^μ undetermined. The vector, q^μ, is thus free to "slide" along the null ray.

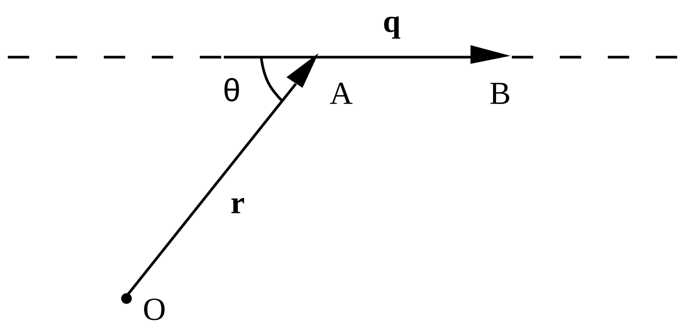

Figure 2:

We can **define** the point of support of the null vector completely by requiring that x^μ also be a null vector. To do this, we have also to consider the scalar product, $S = (x^\mu q_\mu)$. If x^μ and q^μ are both null rays, we have

$$S = x^0 q^0 - x.q = x^0 q^0 (1 - cos\theta)$$

Taking the ratio M/S, we obtain

$$\frac{M}{S} = \frac{x^0 q^0 sin\theta}{x^0 q^0 (1 - cos\theta)} = tan(\theta/2)$$

The angle, θ, between the projections of the null rays in the hyperplane, t = 0, is thus determined, and therefore, a unique null vector, x^μ is also determined.

Unless otherwise specified, we shall use this geometrical model of a twistor pair, i.e., a null ray x^μ, supporting another null ray, q^μ, as shown in Fig. 3. We shall see later that this form allows for additional interpretations, beyond those given in the spinor of the twistor theory.

Let us now show that this matrix pair (τ, q) leads directly to the twistor. To do this, we first put our matrices in spinor notation.

$$(q^\mu \gamma_\mu) = \sqrt{2} \begin{pmatrix} 0 & \bar{q}^{\dot{A}A} \\ q_{A\dot{A}} & 0 \end{pmatrix} \tag{16}$$

$$(x^\mu \gamma_\mu) = \sqrt{2} \begin{pmatrix} 0 & \bar{x}^{\dot{A}A} \\ x_{A\dot{A}} & 0 \end{pmatrix} \tag{17}$$

$$(x^\mu \gamma_\mu)(q^\nu \gamma_\nu) = 2 \begin{pmatrix} \bar{x}^{A\dot{A}} q_{A\dot{B}} & 0 \\ 0 & x_{A\dot{A}} \bar{q}^{A\dot{B}} \end{pmatrix} \tag{18}$$

Writing

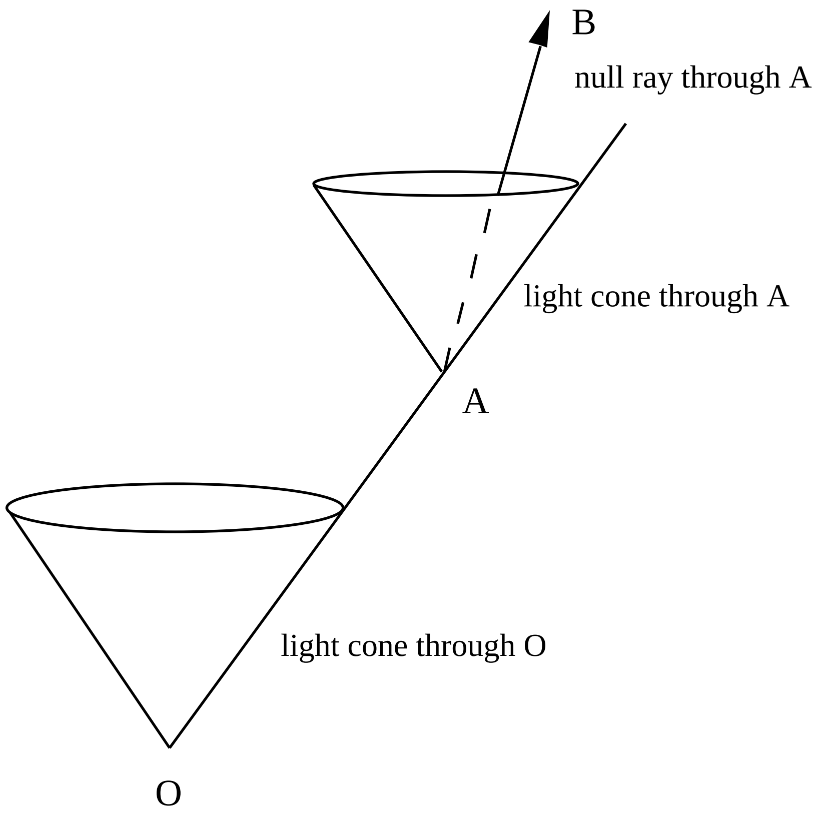
B
null ray through A
light cone through A
A
light cone through O
O

Figure 3:

$$\bar{\tau}^{\dot{A}}_{\dot{B}} = \bar{x}^{A\dot{A}} q_{A\dot{B}}$$

$$\tau^{B}_{A} = x_{A\dot{A}} \bar{q}^{A\dot{B}}$$

we obtain

$$(x^{\mu}\gamma_{\mu})(q^{\nu}\gamma_{\nu}) = 2 \begin{pmatrix} \bar{\tau}^{\dot{A}}_{\dot{B}} & 0 \\ 0 & \tau^{B}_{A} \end{pmatrix} \tag{19}$$

Evidently, $\bar{\tau}^{\dot{A}}_{\dot{B}}$ and τ^{B}_{A} merely repeat the same information in slightly different forms. So to interpret the above matrix, it will be sufficient to consider one of these, for example we write $\tau^{A}_{B} = x^{A\dot{A}} \bar{q}_{\dot{A}B}$. Since q^{μ} is a null ray, we can write $\bar{q}_{A\dot{B}} = \pi_{\dot{A}} \bar{\pi}_{B}$ and so

$$\tau^{A}_{B} = x^{A\dot{A}} \pi_{\dot{A}} \bar{\pi}_{B} \tag{20}$$

The matrix pair (τ, q) then gives rise to

$$(\tau^{A}_{B}, \pi_{\dot{A}}\bar{\pi}_{B}) = (x^{A\dot{A}}\pi_{\dot{A}}\bar{\pi}_{B}, \pi_{\dot{A}}\bar{\pi}_{B}) = (x^{A\dot{A}}\pi_{\dot{A}}, \pi_{\dot{A}})\bar{\pi}_{B} \tag{21}$$

Using equation (1), we then obtain

$$(-i\omega^{A}, \bar{\pi}_{\dot{A}})\bar{\pi}_{B}$$

From the above, it is clear that the twistor is contained in the matrix pair of the Clifford algebra as a kind of "spinor factor". However, as we shall see, the use of the Clifford algebra formulation instead of the twistor formulation will make possible a generalization to a wider range of forms, as well as a natural extension of these forms in such a way that they can be regarded as unfolding from a pre-space. We shall therefore drop the spinor notation from now on (except for certain special purposes) and use the Clifford algebra itself in the expression of our basic theory. In this way, we emphasize that what underlies the twistor concept and what is thus more general than the twistor concept has to do with the Clifford algebra, and not primarily with spinors.

With this in mind, let us now consider how to express the Penrose condition (5) for a null ray as expressed in terms of matrix pairs. We begin by noting that

$$tr(\tau) = \text{a real number}$$

When this is put into the spinor notation, it becomes

$$tr(\tau_B^A) = x^{A\dot{A}}\pi_{\dot{A}}\bar{\pi}_A = \bar{\pi}_A x^{A\dot{A}}\pi_{\dot{A}} = \text{a real number}$$

Writing $x^{A\dot{A}}\pi_{\dot{A}} = -i\omega^A$ we obtain

$$\omega^A\bar{\pi}_A + \bar{\omega}^{\dot{A}}\pi_{\dot{A}} = 0$$

which is the same as the twistor condition (4).

To express the condition for the intersection of two null rays, represented by (τ_1, q_1) and (τ_2, q_2) respectively, we begin with the matrices

$$q_2\tau_1 = \tau_2 x_1 q_1 \qquad \text{and} \qquad q_1\tau_2 = \tau_1 x_2 q_2 \tag{22}$$

(noting that τ_1 and τ_2 are constant along the respective null rays, 1 and 2).

We shall build our intersection condition out of these matrices. But in an algebraic theory, we have to express the invariance of such a condition in terms of a trace, and the above matrices are traceless. In order to obtain matrices that are not traceless, let us bring in an additional matrix $(\frac{1-\gamma^5}{2})Z$ where $Z = (z^\mu\gamma_\mu)$ and z^μ is an arbitrary four-vector factor (a factor $(\frac{1-\gamma^5}{2})Z$ is introduced to make possible a simple connection to the spinor notation, and therefore a comparison with the Penrose condition (8)). This will give us the matrices

$$\lambda_1 = \frac{(1-\gamma^5)}{2}Zq_2\tau_1 = \frac{(1-\gamma^5)}{2}Zq_2x_1q_1$$

$$\lambda_2 = \frac{(1-\gamma^5)}{2}Zq_1\tau_2 = \frac{(1-\gamma^5)}{2}Zq_1x_2q_2$$

Let us now consider

$$tr(\lambda_1) = tr\frac{(1-\gamma^5)}{2}Zq_2x_1q_1$$

Writing $q_{A\dot{A}} = \bar{\pi}_A\pi_{\dot{A}}$ etc, and thus expressing λ_1 in terms of spinor components, we obtain

$$tr(\lambda_1) = \bar{\lambda}_{1\dot{A}}^{\dot{A}} = \pi_{1\dot{A}}\bar{z}^{\dot{A}A}\bar{\pi}_{2A}\pi_{2\dot{B}}\bar{x}_1^{\dot{B}B}\bar{\pi}_{1B} \tag{23}$$

In a similar way, we arrive at

$$tr(\lambda_2) = \bar{\lambda}_{2\dot{A}}^{\dot{A}} = \pi_{2\dot{A}}\bar{z}^{\dot{A}A}\bar{\pi}_{1A}\pi_{1\dot{B}}\bar{x}_2^{\dot{B}B}\bar{\pi}_{2B} \tag{24}$$

We can also write

$$\lambda_{2A}^{A} = \bar{\pi}_{2A} z^{A\dot{A}} \pi_{1\dot{A}} \bar{\pi}_{1B} x_2^{B\dot{B}} \pi_{2\dot{B}} = \pi_{1\dot{A}} z^{A\dot{A}} \bar{\pi}_{2A} \pi_{2\dot{B}} x_2^{B\dot{B}} \bar{\pi}_{1B} \tag{25}$$

This yields

$$tr(\lambda_1) - tr(\bar{\lambda}_2) = \bar{\lambda}_{1\dot{A}}^{\dot{A}} - \lambda_{2A}^{A} = \pi_{1\dot{A}} z^{A\dot{A}} \bar{\pi}_{2A} \pi_{2\dot{B}} (x_1^{B\dot{B}} - x_2^{B\dot{B}}) \bar{\pi}_{1B} \tag{26}$$

(where we have used the hermiticity of $z^{B\dot{B}}$ and $x^{B\dot{B}}$). If the rays intersect, there is a point such that $x_1^{B\dot{B}} = x_2^{B\dot{B}}$. The above is therefore zero for all points, x_1^{μ} and x_2^{μ} along the rays in question. And so, we obtain the Penrose intersection condition (8). We have thus completed the demonstration that the basic twistor relationships can all be expressed in terms of the Clifford algebra. Let us now show how space-time displacements can also be expressed in terms of this algebra. To do this, we first note that a displacement, Δx^{μ}, leads to

$$\Delta\tau = (\Delta x^{\mu}\gamma_{\mu})(q^{\mu}\gamma_{\mu}) \tag{27}$$

We now go on to an arbitrary function of the matrix pair, $F(\tau_{ab}, q_{cd})$ where τ_{ab} and q_{cd} are an explicit expression of the components of the matrices, τ and q. The infinitesimal displacement operation is the

$$\Delta F = \Delta\tau_{ab}\frac{\partial F}{\partial \tau_{ab}} = (\Delta x^{\mu}\gamma_{\mu})_{ab}(q^{\mu}\gamma_{\mu})_{cd}\frac{\partial F}{\partial \tau_{ab}} \tag{28}$$

We represent such a displacement in Fig. 4. The vector, q^{μ}, is displaced into a parallel and equal vector, q'^{μ}. We can regard this displacement as brought about by a **vector couple** This consists of a vector, q^{μ} and another one, $-q^{\mu}$. The net effect of this couple is to cancel q^{μ} and replace it by q'^{μ}. Such a couple can be represented by a special matrix pair.

$$(\tau + \Delta\tau, q) - (\tau, q) = (\Delta\tau, 0) \tag{29}$$

Any matrix pair having zero for its q matrix clearly represents a vector couple. From equation (27), it follows that this couple is invariant to a space-time displacement. Such couples will be important in the description of properties of geometric structures (e.g., angular momentum), whose meaning is invariant to a change of origin of the coordinate system. This kind of displacement invariance generalizes the notion of Lorentz invariance of properties of structures that do not depend on the Lorentz frame. That is to say, in this generalized twistor theory, we seek invariants of the entire Poincare group, and some of these may be buiIt out of vector couples. This notion will be extended further as we go along.

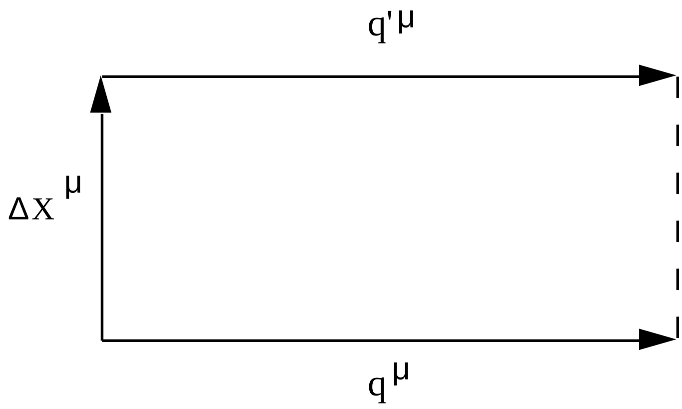

Figure 4:

4 Some further geometrical possibilities in the Clifford algebra

Weshallnowshowofthefurther geometricalforms thatarepossibleinterpretations of the Clifford algebra. Firstly, the fact that the Clifford algebra contains a bounded null vector, q^μ, and not just a null ray, immediately allows us to give an additional interpretation to the formalism. To see how this comes about, we note that in the matrix $\Delta\tau = (\Delta x^\mu \gamma_\mu)(q^\mu \gamma_\mu)$. Δx^μ and q^μ have interchangeable roles. There fore, if we regard $\Delta\tau$ as determined by multiplication from the right (rather than from the left), we are led to define the matrix pair

$$(\Delta\tau, \Delta x) \tag{30}$$

Whereas(as showninFig. 4)our originaldefinition isinterpreted asadisplacement of the vector q^μ by Δx^μ, we can now use the matrix pair notation to represent a displacement of Δx^μ by q^μ, as shown in Fig. 5. The two displacements together determine a parallelogram of null vectors, as shown in Fig. 6. So the property of parallelism is expressed in terms of purely algebraic relationships. Without specification of a bounded vector, q^μ , which is part of the algebra, this would not have been possible. This is a further example of how geometric relationships are contained in a Clifford algebra (capable, of course, of generalization to the implicate order). We now go on to express more complex geometric relation ships in this way. To do this, let us consider ordered sequence of null vectors.

$$q_1^\mu, q_2^\mu ......q_n^\mu..... \tag{31}$$

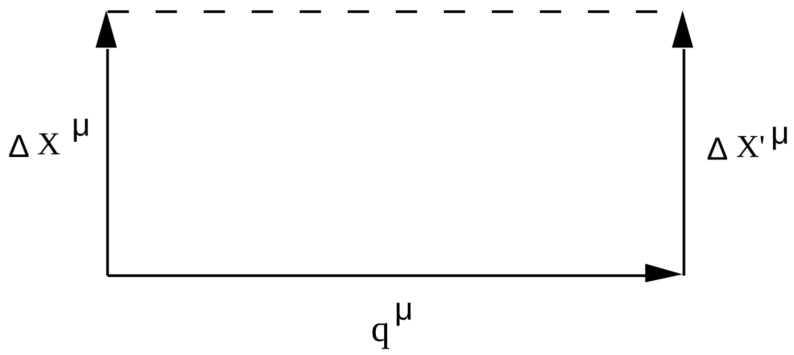

Figure 5:

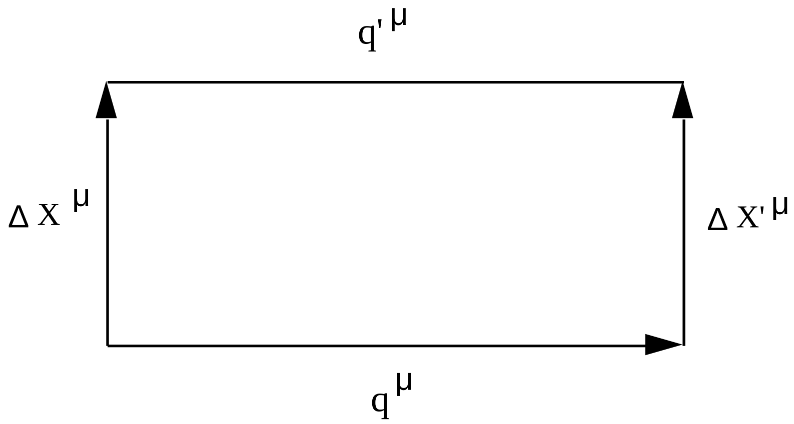

Figure 6:

We begin with the first two of this sequence by introducing the notation τ_{12} which expresses the two vectors that make it up

$$\tau_{12} = q_2 q_1$$

These give the matrix pairs

$$(\tau_{12}, q_2) \qquad \text{and} \qquad (q_2, \tau_{12})$$

depending on whether we use left or right multiplication. We shall, however, in the rest of this paper restrict ourselves to left multiplication.

We then define

$$\tau_{123} = q_3 q_2 q_1 \tag{32}$$

with its twistor triplet $(\tau_{321}, \tau_{21}, q_1)$

A simple calculation shows that

$$\tau_{321} = V + M \tag{33}$$

where

$$V = q_3(q_1^\mu q_{2\mu}) + q_2(q_3^\mu q_{1\mu}) + q_1(q_2^\mu q_{3\mu}) \tag{34}$$

and

$$M = \left[\frac{q_1^\mu q_2^\nu q_3^\rho}{3!}\right] \gamma_\mu \gamma_\nu \gamma_\rho \tag{35}$$

where $(q_l^\mu q_2^\nu q_3^\rho)$ is the totally antisymmetrized tensor.

To show what this means, let us consider the projections of all rays in the hyperplane, t = 0. If q_3^μ represents a ray from the origin, while q_2^μ is attached to the end of q_3^μ, and q_1^μ to the end of q_2^μ, we see that $q_1^\mu, q_2^\mu, q_3^\mu$ determine a triangle ABC , of sides having dimensions AB = $|q_2|$ and BC = $|q_1|$. The higher moment $M^{\mu\nu\rho} = [q_3^\mu q_2^\nu q_1^\rho]$ determines the volume of a tetrahedron based on ABC, whose apex is at the origin. This volume is a kind of extension of the idea of moment to the triangle ABC. Such a moment determines only the **plane** in which ABC lies, and not the location of ABC itself. Thus, if only the moment is given, the triangle ABC is freely displaceable on this plane(corresponding to our previous result that q_1^μ is a vector that is freely displaceable along the line of the null ray).

To locate this triangle in its plane we need V. Evidently V determines a point, which is a kind of barycenter of the triangle, with each vector weighted according

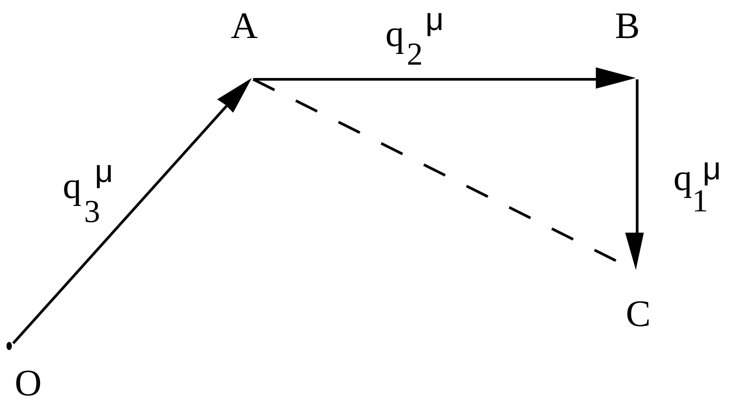

Figure 7:

to the Minkowski scalar product of the rays corresponding to the two ends of the opposing sides of the triangle.

The twistor triplet $(\tau_{321}, \tau_{21}, q)$ therefore determines the triangle. Parts of the triplet such as (τ_{321}, τ_{21}) can easily be shown to determine the moments of vectors around other points in the geometric figure.

As we include higher matrices, such as τ_{4321}, etc, we will describe elements of higher dimension. In an indefinite chain going beyond q_4, we can interpret any four consecutive terms in terms of geometric elements, including lines, planes, and hyper volumes. The further terms in the sequence can be regarded as defining a chain of structures (e.g., as developing in time, or as extended in space).

We shall not, however, pursue the point further here, as it is not our purpose in this paper to give a detailed analysis of how geometrical structures are contained in the Clifford algebra. (We hope to do this in a later paper.) What we wish to emphasize is only the following:

(i) The Clifford algebra describes a wide range of geometric forms and their relationships.

(ii) ln the unlimited matrix chain, we may make a transformation, $q^\mu \rightarrow q^\mu + \Delta q^\mu$ at any point in the chain. This will describe a displacement of the entire part of the space that follows the "break" (in the order in which the transformations operate). So the displacement operation can be given a "local" significance. That is, some part of the structure can be displaced independently of nearby parts, in a way that corresponds roughly to a "local" transformation (i.e., one that can be mapped into a local "tangent space"). This sort of transformation is necessary, if structures in different parts of space-time are to have the kind of indepen dence that is needed, to build up the right general properties of space as we know them.

(iii) We have interpreted a elements of the Clifford algebra in terms of geometrical structures and their relationships.

(iv) lf we derive structures by left multiplication, we form a left ideal. We can also treat the matrix sequence in the opposite order, and then we derive structures by right multiplication form ing a right ideal. The two ideals together give rise to further structural relationships (as appeared with the parallelogram de rived from right and left multiplication of a twistor chain of the elements). This gives rise to a natural way of understanding duality, in terms of left and right ideals in the algebra.

All this richness of geometric structure can be put into the implicate order, because it canundergo anarbitrary inner automorphism. To obtain avivid image of what this means,we could make a hologram of the structures, and all the geometric relation ships would still be contained implicitly in it, although in a form that was enfolded in the total space.

5 Non-null vectors expressed through the Clifford algebra

A further extension of the twistor concept is to bring in non-null vectors through the Clifford algebra. Indeed, in the definition of

$$\tau = xq = 1\frac{(x^\mu q_\mu)}{2} + \frac{(x^\mu q^\nu - x^\nu x^\mu)}{2}\gamma_\mu\gamma_\nu$$

there is no need to restrict either q^μ or x^μ to null vectors. The moment $M^{\mu\nu} = (x^\mu q^\nu - x^\nu q^\mu)$, along with q itself, still determines a line, through its projection in the hyperplane t = 0, and through itsintercept, inthe same wayas happens with null vectors. There is, however, one further new feature in this case. For a movement along this line, represented by $\Delta x^\mu = \alpha q^\mu$, the scalar product, $S = (x^\mu q_\mu)$ is no longer unaffected. Therefore a unique vector, x^μ is determined by a matrix τ. This means that the matrix pair, (τ, q) no longer determines a line directly, but rather, it has to determine a vector q^μ, with its point of support at the end of x^μ.

To determine only the line, we need to use the matrix

$$M = \frac{(x^\mu q^\nu - x^\nu q^\mu)}{2}\gamma_\mu\gamma_\nu = \frac{M^{\mu\nu}}{2}\gamma_\mu\gamma_\nu = xq - qx \qquad (36)$$

which is based on the moment of q^μ. Indeed, this moment is just what remains constant when any vector, x^μ, touching the line is varied. So the matrix pair (M, q) is what determines the line (for the special case of the null ray, the scalar product is also independent of this point, and so, we can let the line be determined instead by τ). The condition for the intersection of two lines is a generalization of equation (26). One can show by a calculation similar to that leading to (26) that if

$$\alpha_1 = \frac{(1-\gamma^5)}{2} Z q_2 M_1 q_1$$

$$\alpha_2 = \frac{(1-\gamma^5)}{2} Z q_1 M_2 q_2$$

then the lines corresponding to M_1 and M_2 will intersect if and only if

$$tr(\alpha_1) - tr(\alpha_2) = 0 \tag{37}$$

6 The Clifford algebraic description of a helical path

Let us now consider the use of the Clifford algebra to express how vectors of different magnitudes and directions and points of support can be put together, to make up an arbitrary path. To orient ourselves, let us imagine a Brownian type curve made up of small directed line segments. We allow steps backward in the time as well as forwards, because we are discussing geometry, and not paths of entities as particles. (Equally, we could interpret this in terms of Feynman's idea that a step backward in time is equivalent to an anti-particle.) In general, the constituent vectors need not be null rays, as the matrix pair notation allows for the description of arbitrary vectors.

We represent such an irregular line by Fig. 8. We let Δx_n^μ represent the n^{th} step, while x_n^μ is the vector from the origin to the beginning of the n^{th} step. We have, for the matrix pair representing the nth step (τ_n, $\Delta x_n)$ with

$$\tau_n = \frac{S_n}{2} 1 + \frac{M^{\mu\nu}}{2} \gamma_\mu \gamma \nu$$

$$S_n = (x_n^\mu \Delta x_{n\mu})$$

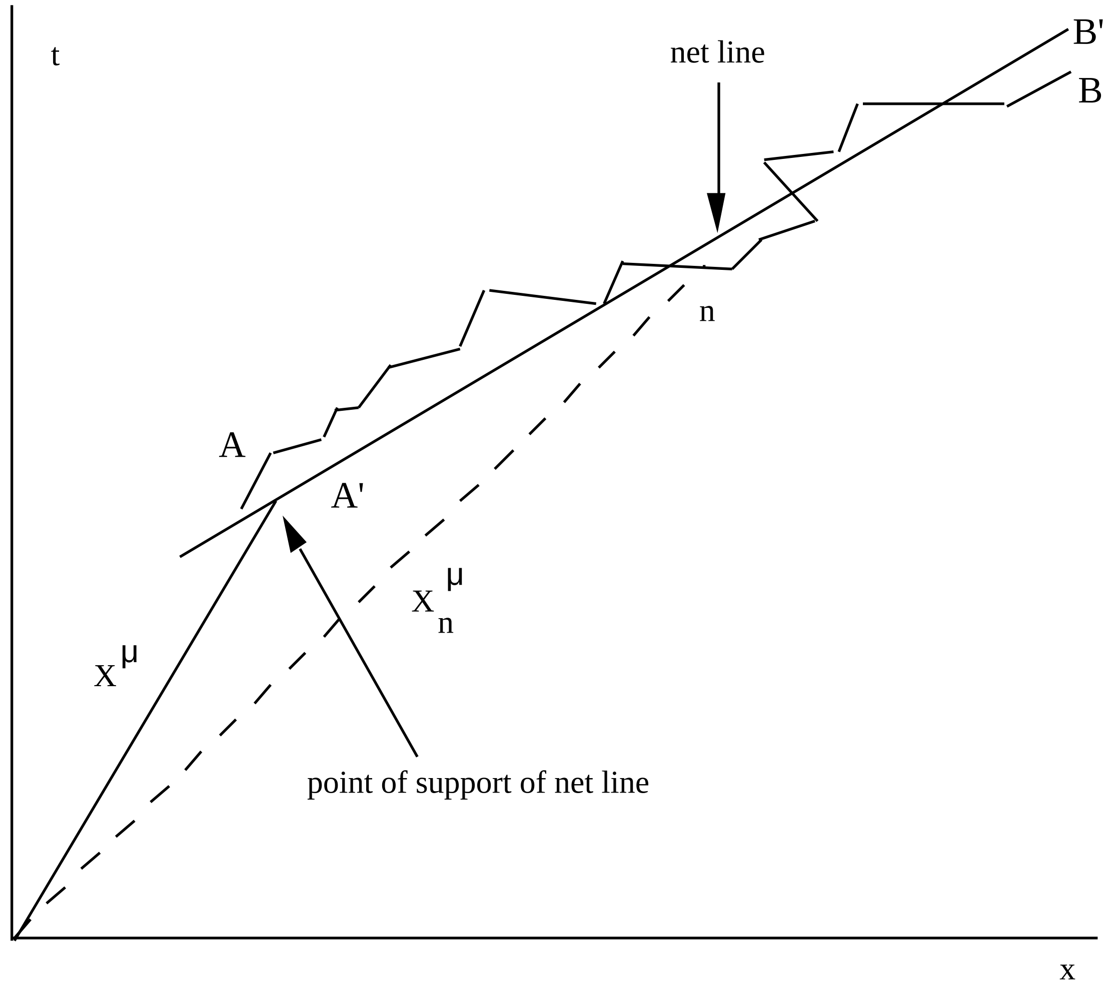

Figure 8:

$$M_n = x_n^\mu \Delta x_n^\nu - x_n^\nu \Delta x_n^\mu$$

What we used to do is to replace the irregular Brownian motion type of curve by a single line that is equivalent in certain key respects. These are:

(a) The net vector, Δx^μ must be the sum of the constituent vectors, Δx_n^μ i.e.

$$\Delta x^\mu = \sum_n \Delta x_n^\mu \tag{38}$$

(b) The moment of the equivalent line must be the same as the sum of the moments, $M_n^{\mu\nu}$; i.e.

$$M^{\mu\nu} = \sum_n (x_n^\mu \Delta x_n^\nu - x_n^\nu \Delta x_n^\mu) \tag{39}$$

(c) The point of support, x^μ of the net vector must be such that the net scalar product, S, is the sum of the scalar products S_n; i.e.

$$(x^\mu \Delta x_\mu) = \sum_n (x_n^\mu \Delta x_n^\mu) \tag{40}$$

Firstly, we note that if the net vector, Δx^μ is not a null vector, then the addition of a displacement, $\delta x = \alpha x^\mu$, along the net line will add a non-zero quantity, $\alpha(x^\mu x_\mu)$ to the scalar product. So α can always be chosen to give rise to a position, $x_0^\mu + \alpha x^\mu$; in which the condition (3) on the scalar products is satisfied (we discuss the special case that Δx^μ is a null vector further on).

Since the satisfaction of condition (1) is straightforward, there remains only the question of condition (2) to discuss. The problem of satisfying this condition is very similar to that of summing up forces that are in different lines in three dimensions. If we let F_n represent one of the forces, and r_n the (three dimensional) vector connecting the origin to this force, then we may define the total force as $F = \sum_n F_n$ and the total momentum $M = \sum_n r_n \times F_n$

There is a well-known theorem that there is a unique line, such that M is the moment of the net force, F acting along this line, plus a couple, C (i.e., a freely displaceable torque) round the direction of the line itself. Clearly, the scalar product, $F.C$, corresponds to a kind of helicity, or "twist" in the structure of the net force (at least up to a factor of proportionality). This situation is represented in Fig. 9.

F represents the net force, which is on a line perpendicular to BA. That part of the net moment which is perpendicular to F determines the line along which

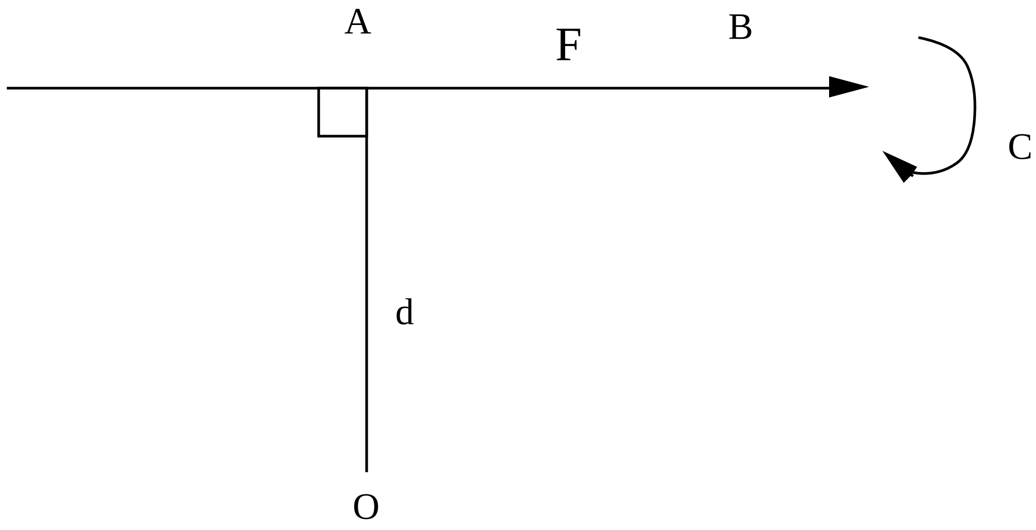

Figure 9:

the net force lies (i.e., it determines the perpendicular distance, d, to the origin O). The force is, of course, freely displaceable. The couple, C, is shown by the curved arrow, which maybe clockwise or counter-clockwise, whenviewed inthe direction of the force. For our problem with four-dimensional lines, we have already seen in Section 5 that the net moment $M^{\mu\nu}$ along with the net vector Δx^{μ} determine a net line uniquely, provided that this moment is entirely in a two-dimensional element containing Δx^{μ} itself. If we project into the hyperplane t = 0, this means that the moment vector, M, whose space components are $M_i = \varepsilon_{ijk} M_{jk}$, is perpendicular to the line. If the net moment contains an additional component C parallel to the line, then, as happens with forces, the net vector, Δx^{μ} must have a corresponding couple, $C^{\mu\nu}$, added to it. So, in general, an irregular curve may be summed up as equivalent to a certain vector Δx^{μ} ; on a point of support, x^{μ}, plus a couple, $C^{\mu\nu}$. The corresponding matrix pair is (τ,Δx), where τ is given by

$$\tau = (x^{\mu}\gamma_{\mu})(\Delta x^{\nu}\gamma_{\nu}) + \frac{c^{\mu\nu}}{2}\gamma_{\mu}\gamma_{\nu} \tag{41}$$

For the case in which Δx^{μ} is a null ray, a different treatment is needed. For here, the point of support of Δx^{μ} is not determined by the matrix pair but is freely moveable (unless some other requirement, such as x^{μ} is a null ray, is added). In this case, it will not, in general, be possible to satisfy condition (3) on the scalar product. Indeed, for a null ray, this scalar product is already determined by the ray alone (e.g., by evaluat ing it at the intercept, (0 ,ξ)). So, we shall therefore drop the requirement of satisfying condition (3), and will restrict our selves to satisfying (1) and (2) only. Here, we recall that for a null ray, the entire matrix is constant along its line. Since the scalar product contains no information that is not contained in the matrix pair $(M, \Delta x)$ (where $M = \frac{M_{\mu\nu}}{2}\gamma_\mu\gamma_\nu$), we may equally well replace this latter by the pair (τ , Δx) as a specification of the null ray.

The only difference in the procedure for null rays is thus basically just to ignore the net scalar product, and so,to refrain from determining the point of support of Δx^μ . The theorem that the net moment is equiva!ent to a net vector along its ray plus a net couple around the direction of this ray still holds. Δx^μ is a null vector, we can go back to the twistor formulation for the part of the matrix pair (τ,Δx). However, to this must be added the couple $(C, 0)$.

We see then that whether the resultant vector, Δx^μ is null or not, we are able to sum up an arbitrary curve as a net vector, plus a kind of "twist" around the direction of this vector. Of course, this twist will in general be arbitrary and highly irregular. We may, however, consider a situation in which, at least statistically, the average couple per unit length will be a constant. This will constitute a line with a systematic twist,which may be thought of as a kind of helix.

We have thus described lines with an intrinsic twist or helicity in terms of matrix pairs. As we have pointed out earlier, however, in the spinor approach, such twist or helicity is obtained only in an indirect reflection of the line in a family of other lines (i.e., the Robinson congruences). On the other hand, in the algebraic approach, the line will give rise to an intrinsic "angular momentum" (e.g., if a particle of mass m moves along it, the average angular momentum will be proportional to $\overline{(\gamma^\mu \Delta x^\nu - \gamma^\nu \Delta x^\mu)}$. Since this contains a couple, we have a possibility of an intrinsic spin, which is invariant to space-displacement. This is a good basis for a model of "spin" of elementary particles of zero rest mass, since it has just the right properties, including an intrinsic chirality.

lf we further supposed that the average couple was a continuous function of space and time, we could require, in harmony with special relativity, that the distribution of lines in one region be a Lorentz transform of that in other regions. However, we could not have different chiralities, if such continuity is to be maintained. Thus, it becomes possible for space to have a definite chirality, which would be the ground of parity non-conservation in β-decay. This is a kind of symmetry-breaking, in the sense that once the lines in any given system have a certain twist, they will favor a similar twist in neighbouring regions.

Theuseofamatrixpairtobuildanetlineoutofconstituentlinesalsoestablishes a fundamental connection between the rotation group and the translation group. In particular, the moment of the "twist" is, as we have seen, proportional to an "angular momentum" direction of the ray. The associated rotations (generalized to Lorentz transformations) that leave the ray in variant are just the "little group" as treated in standard relativistic quantum theory. But now, we see the significance of the little group as denoting a certain invariance of the chiral structure of a line made up of small constituent lines. This opens the way to further physical interpretation of the geometrical structures associated with the Dirac equation.

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