# From Sets to Clifford Algebras*

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## 1 Sets, Reflections, Subsets, and XOR.

Start with von Neumanns Set Theoretical definition of the Natural Numbers N:

$$0 = \emptyset, \quad 1 = \{\emptyset\}, \quad 2 = \{\emptyset, \{\emptyset\}\}, \ldots, \quad n+1 = n \bigcup \{n\}.$$

in which each Natural Number $n$ is a set of $n$ elements. By reflection through zero, extend the Natural Numbers $N$ to include the negative numbers, thus getting an Integral Domain, the Ring $Z$. Now, following the approach of Littlewood [1], consider a set

$$S_n = \{1, e_1, e_2, \ldots, e_n\} \quad \text{of } n+1 \text{ elements.}$$

Consider the set $2^{S_n}$ of all of its $2^n$ subsets. These will be of the form

$$1, \{e_j\}, \{e_i e_j\}, \{e_i e_j e_k\} \ldots \{e_1 e_2 \ldots e_n\}$$

with a product on $2^{S_n}$ defined as the symmetric set difference XOR. An example of an XOR product is as follows. If $A = \{2, 4, 5, 8\}$ and $B = \{4, 5, 6, 8\}$ then the XOR product is $A \ominus B = \{2, 6\}$ In more general terms denote the elements of $2^{S_n}$ by $m_A$ where $A$ is in $2^{S_n}$.

## 2 Discrete Clifford Algebras.

To go beyond set theory to Discrete Clifford Algebras, enlarge $2^{S_n}$ to $DClG(n)$ by: order the basis elements of $S_n$, and then give each element of $2^{S_n}$ a sign, either +1 or -1, so that $DClG(n)$ has $2^{n+1}$ elements. This amounts to orientation of the signed unit basis of $S_n$.

*Based on a paper by Tony Smith[2].

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Then define a product on $DClG(n)$ by

$$(x_1 e_A)(x_2 e_B) = x_3 e_{A \ominus B}$$

where $A$ and $B$ are in $2^{S_n}$ with ordered elements, and $x_1, x_2$, and $x_3$ determine the signs.

For given $x_1$ and $x_2, x_3 = x_1 x_2 x(A, B)$ where $x(A, B)$ is a function that determines sign by using the rules $e_i e_i = +1$ for $i$ in $S_n$
and $e_i e_j = -e_j e_i$ for $i \neq j$ in $S_n$ ,

then writing $(A, B)$ as an ordered set of elements of $S_n$, then using $e_i e_j = -e_j e_i$ to move each of the $B$-elements to the left until it: either meets a similar element and then cancelling it with the similar element by using $e_i e_i = +1$ or it is in between two $A$-elements in the proper order. $DClG(n)$ is a finite group of order $2n + 1$.

It is the Discrete Clifford Group of n signed ordered basis elements of $S_n$. Now we can construct a discrete Group Algebra of $DClG(n)$ by extending $DClG(n)$ by the integral domain ring $Z$ and using the relations

$$e_i e_j + e_j e_i = 2\delta(i, j)1$$

where $\delta(i, j)$ is the Kronecker delta.

Since $DClG(n)$ is of order $2n+1$, and since two of its elements are -1 and +1, which act as scalars, the discrete Group Algebra of $DClG(n)$ is $2n$-dimensional. The vector space on which it acts is the $n$-dimensional hypercubic lattice $Z_n$.

The discrete Group Algebra of discrete Clifford Group $DClG(n)$ is the discrete Clifford Algebra $DCl(n)$.

## 3 Examples

Here is an explicit example showing how to assign the elements of the Clifford Group to the basis elements of the Clifford Algebra Cl(3): First, order the $2^{3+1} = 16$ group strings into rows lexicographically:

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

Then discard the first bit of each string, because it corresponds to sign which is redundant in defining the Algebra basis. This reduces the number of different strings to $2^3 = 8$:

000
001
010
011
100
101
110
111

Then separate them into columns by how many 1s they have:

| | | | |
|---|---|---|---|
| 000 | | | |
| | 001 | | |
| | 010 | | |
| | | 011 | |
| | 100 | | |
| | | 101 | |
| | | 110 | |
| | | | 111 |

Now they are broken down into the 1331 graded pattern of the Clifford Algebra Cl(3).

## References


[1] Littlewood, J. Lond. Math. Soc. **9** (1934) 41-50.

[2] Smith, F.,D., From sets to Quarks, hep-ph/9708379.