

Clifford Algebras in Symplectic Geometry and Quantum Mechanics

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Accepted. To appear in *Fond. Phys.*

arXiv:1112.2378 (math-ph)

DOI:1007/s10701-012-9634-z

Abstract

The necessary appearance of Clifford algebras in the quantum description of fermions has prompted us to re-examine the fundamental role played by the quaternion Clifford algebra, $C_{0,2}$. This algebra is essentially the geometric algebra describing the rotational properties of space. Hidden within this algebra are symplectic structures with Heisenberg algebras at their core. This algebra also enables us to define a Poisson algebra of all homogeneous quadratic polynomials on a two-dimensional sub-space, \mathbb{F}^a of the Euclidean three-space. This enables us to construct a Poisson Clifford algebra, \mathbb{H}_F , of a finite dimensional phase space which will carry the dynamics. The quantum dynamics appears as a realisation of \mathbb{H}_F in terms of a Clifford algebra consisting of Hermitian operators.

1 Introduction

In a series of lectures at the annual Askloster Seminars, hosted by Georg Wikman, one of us (BJH) developed a new approach to quantum mechanics based on the notion of ‘structure process’, which provides a very different way of looking at quantum phenomena [9]. There it was shown how orthogonal Clifford algebras arise from a collection of what we call ‘elementary processes’. It should be no surprise that Clifford algebras play an essential role in quantum mechanics since these algebras are necessary for a description of both the Pauli and Dirac particles. The fact that the Schrödinger particle can also be described in terms of a Clifford algebra may come as a surprise.

In this paper we will not discuss these elementary processes in their full generality. Rather we will take these ideas to motivate a discussion of the role of quaternions in bringing quantum mechanics and geometry closer together. Another motivation for our present discussion comes directly from the work of Binz and Pods [2]. This shows that a natural symplectic structure can be constructed using the quaternions. This, in turn, gives rise to the Heisenberg

algebra and it is the appearance of this algebra as a geometric feature that enables us to make contact with quantum mechanics.

We start by constructing the skew-field of quaternions, a Clifford algebra. We call this algebra, together with the Clifford algebra of an Euclidean plane, the elementary Clifford algebra. Combining this structure with the complex numbers (which drop out of the real quaternions), the general theory of real Clifford algebras as presented in Greub [7], then allows us to construct the (finite-dimensional) Clifford algebras of bilinear forms of arbitrary index out of the elementary ones, thus enabling us to discuss the Schrödinger, Pauli and Dirac particles in one geometric structure.

In quantum mechanics the symplectic geometry plays a fundamental role through the Heisenberg group as has been shown by de Gosson [4]. How is this structure hidden in Clifford algebras? To answer this question we will focus, in particular, on the geometric aspects of the quaternion Clifford algebra which encodes a two-dimensional *symplectic space* \mathbb{F}^a , characterised by selecting a particular pure quaternion a . \mathbb{F}^a is the tangent plane to the two-sphere at a . The choice of a is arbitrary. We could have chosen another pure quaternion, say b , and this will give rise to a different two-dimensional symplectic space \mathbb{F}^b . However these pure quaternions are related by the Clifford group $SU(2)$, so that we have an infinity of Heisenberg algebras within the quaternion Clifford, $\mathbb{H} = \mathbb{R} \cdot e \oplus \mathbb{E}$, with \mathbb{E} being the Euclidean three-space.

Each symplectic space gives rise to a two-dimensional classical *phase space*, while the whole structure gives rise to a $2n$ -dimensional phase space, which in turn, gives rise to a Poisson algebra. We then find the Poisson algebra of all homogenous quadratic polynomials over a two-dimensional *phase space* determines a natural Clifford algebra as well, which we will call the *Poisson Clifford algebra* of the phase space. This is isomorphic to the geometric Clifford algebra. The isomorphism is based on the construction of Hamiltonian vector fields. The Poisson algebra of homogenous quadratic polynomials of any phase space of dimension higher than two is the linear image of a natural subspace of the Poisson Clifford algebra. Finally we introduce a realization of the characteristic Clifford algebra of a phase space generated by Hermitian operators. This realization is based on the quantization map of the above mentioned Poisson algebra. This final step is where we make contact with quantum mechanics. This elementary presentation follows essentially the book “The Geometry of Heisenberg Groups” [2].

2 From Processes to Clifford algebras

2.1 Processes

In Hiley [9] the notion of an elementary process was introduced. A set of these elementary processes can then be ordered by defining a simple multiplication rule, which then produces a structure that is isomorphic to the quaternion Clifford algebra. A detailed discussion of the physical motivation and the underlying philosophical grounds for introducing the fundamental notion of a *process* and its history, we refer to Hiley [9], Kauffman [13] as well as to Penrose [15].

The fundamental idea behind this approach is to show how a geometric picture of an Euclidean three-space, \mathbb{E} , emerges from the order of the elementary processes encapsulated in the algebra. We can represent each *process* as a *directed* one-simplex $[P_1P_2]$, connecting the opposite distinguishable poles of the process P_1 and P_2 . As a basic assumption, we regard these process as *indivisible*, meaning that the poles are merely distinguishing features of the process that cannot be separated from the process itself. Since we are assuming our processes are directed, we assume $[P_1P_2] = -[P_2P_1]$. The totality of one-simplexes forms a real linear space isomorphic to the rotational symmetries of \mathbb{E} .

An *inner multiplication* for the one-simplexes is defined by

$$[P_1P_2] \bullet [P_2P_3] = [P_1P_3]. \quad (1)$$

This multiplication rule encapsulates the Leibnizian notion of the order of succession.

Starting with the three elementary processes $[P_0P_1]$, $[P_0P_2]$ and $[P_1P_2]$ we find

$$\begin{array}{c|ccc} 1 & [P_0P_1] & [P_0P_2] & [P_1P_2] \\ \hline [P_0P_1] & -\mathbf{e} & -[P_1P_2] & [P_0P_2] \\ [P_0P_2] & [P_1P_2] & -\mathbf{e} & -[P_0P_1] \\ [P_1P_2] & -[P_0P_2] & [P_0P_1] & -\mathbf{e} \end{array} \quad (2)$$

In this structure $[P_0P_0]$, $[P_1P_1]$ and $[P_2P_2]$ are easily shown to act as two sided unities, which we identify and denote by a single unity \mathbf{e} . In this way we have generated a group from the elementary processes. We can quickly identify this group by making the following notation change $\mathbf{i} := [P_0, P_1]$, $\mathbf{j} := [P_0, P_2]$ and $\mathbf{k} := [P_1, P_2]$. In this way we immediately see that $\mathbf{e}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ generate the quaternion group. The resulting skew-field \mathbb{H} of *quaternions* is a real *Clifford algebra* [7], which can be verified by the defining universal property presented below. The quaternions are often called a hypercomplex system.

In [9] it was shown how to generalise the process algebra to include the Pauli- and Dirac-Clifford algebras. This has enabled Hiley and Callaghan [12] [10] [11] to apply these algebras in a novel way to the quantum mechanics of the Schrödinger, Pauli and Dirac particles. This has led to a systematic generalisation of the Bohm approach [3] to the relativistic domain.

Our approach should not be confused with the attempt made by Finkelstein *et al.* [5] to develop a quaternion quantum mechanics. These authors were investigating the possibility of using a Hilbert space over the quaternion field. Our approach is exploring a very different structure.

2.2 Clifford Algebras

For convenience we repeat the notion of a Clifford algebra by means of its *universal property*. Let \mathcal{A} be an associative algebra with a unit element e_A . A *Clifford map* from the \mathbb{R} -linear space \mathbb{F} to \mathcal{A} is a linear map φ which satisfies

$(\varphi(h))^2 = b(h, h)e_A$. Here b is an \mathbb{R} -valued, *symmetric bilinear* map defined on \mathbb{F} . The *universal property* mentioned above reads as follows:

A Clifford algebra C_F over \mathbb{F} is an associative algebra C_F with unit element e , together with a *Clifford map* i_F from \mathbb{F} to C_F subject to the following two conditions: $i_F(F) \subset C_FF$ generates C_F and to every Clifford map φ from \mathbb{F} to C_F , there is a unique *algebra homomorphism* Φ from C_F to \mathcal{A} satisfying $\varphi = \Phi \circ i_F$.

\mathbb{H} is a very basic Clifford algebra as we will see in the next section. The bilinear map is $-\langle, \rangle$, the negative of a scalar product \langle, \rangle on a two-dimensional subspace \mathbb{F} in \mathbb{E} (cf. [7]).

Let us point out at this stage that the \mathbb{R} -algebra $\text{End } F$ with the *composition* as its product is a Clifford algebra. It is determined by a scalar product \langle, \rangle on \mathbb{F} (cf. [7]). The multiplication table for $J, A, B \in \text{End } F$ with respective matrices

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

is

| | | | | |
|----|-----|----|----|-----|
| id | J | A | B | (4) |
| J | -id | -B | A | |
| A | B | id | J | |
| B | -A | -J | id | |

This is due to

$$A \circ J = B, \quad J \circ B = A, \quad A \circ B = J, \quad \text{and} \quad [A, B] = 2 \cdot J \quad (5)$$

as a direct calculation shows. (See Binz and Pods [2]). Here $[A, B]$ is the *commutator* of A and B .

This multiplication table (4) differs only in the signs down the diagonal from table (2) and so can be regarded as arising from the structure process in \mathbb{F} . In this case they are isomorphic to the Clifford algebra $\text{End } F$.

In fact, any real finite-dimensional Clifford algebra of a symmetric bilinear form of arbitrary index is a finite graded tensor product of the Clifford algebras \mathbb{C} , \mathbb{H} and $\text{End } F$ (cf. [7]). In this sense the processes introduced above determine all these Clifford algebras.

We leave the algebraic treatment of processes and focus on another concept within real Clifford algebras. We will show how these algebras contain symplectic spaces, a basic entity of Hamiltonian mechanics and quantum mechanics. Of course, these two approaches to Clifford algebras hint at a relation between processes that could give rise to symplectic geometry, to Hamiltonian mechanics and to quantum mechanics. The latter relations are discussed in Hiley [9].

3 Clifford Algebras of Symplectic Spaces

3.1 The geometric Clifford Algebra of a symplectic space

In this subsection we will show how a characteristic Clifford algebra emerges from each (*2n-dimensional*) symplectic space.

To this end we start with \mathbb{F} considered as a *two-dimensional*, real linear space equipped with a (constant) non-degenerated, skew-symmetric bilinear two-form ω , a *symplectic form*. Hence \mathbb{F} is a two-dimensional *symplectic space* (cf. [4]). In two dimensions, ω serves a *volume form* on \mathbb{F} . In other words \mathbb{F} is an *oriented*, two-dimensional linear space (cf. [4]).

Let us analyse this assumption in a little more detail. For simplicity we choose a scalar product \langle, \rangle on \mathbb{F} and represent ω by a *skew-adjoint* isomorphism S so that

$$\omega(v, w) = \langle S(v), w \rangle \quad \forall u, v \in \mathbb{F}. \quad (6)$$

There is an orthonormal basis in the Euclidean space, \mathbb{F} , such that S has the form

$$S = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}; \quad \kappa \in \mathbb{R}.$$

Without loss of generality, we may choose $\kappa = 1$ if \langle, \rangle is rescaled suitably; in this case we write J instead of S and its matrix is

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (7)$$

as represented in (3). This map gives rise to an *almost complex structure*.

Therefore, we conclude that, given a symplectic space \mathbb{F} with skew form ω , there is some scalar product and an isomorphism J satisfying

$$J^2 = -id \quad (8)$$

representing ω in the sense of (6).

Vice versa, given ω and some J fulfilling (8), there is a *unique* scalar product \langle, \rangle , given by

$$\langle v, w \rangle := \omega(J(v), w) \quad \forall u, v \in \mathbb{F}. \quad (9)$$

In fact, $J \in Sp(F)$, the group of all isomorphisms of \mathbb{F} preserving the symplectic form. This group is called the *symplectic group*.

To construct a *phase space* out of \mathbb{F} we have to specify a *configuration space*, i.e. the space of positions in \mathbb{F} . This space is a real line $\mathbb{R} \cdot e_1 \subset \mathbb{F}$ spanned by the vector $e_1 \in \mathbb{F}$, say. Then the line $\mathbb{R} \cdot J(e_1) \subset \mathbb{F}$ represents the space of *momenta*. \mathbb{F} decomposed in this way is a *phase space*. For simplicity of notation let $e_2 := J(e_1)$. The basis $\{e_1, e_2\} \subset \mathbb{F}$ can be assumed to be symplectic, i.e. $\omega(e_1, e_2) = 1$. We will denote the coordinates by q and p , respectively. The symplectic form ω on the phase space \mathbb{F} is then the *canonical* one (cf. [14]). Hence

Lemma 1 *Given a symplectic space \mathbb{F} with symplectic form ω and some almost complex structure $J \in Sp(F)$ with $J^2 = -id$, there is a unique scalar product \langle, \rangle defined by*

$$\langle v, w \rangle = -\omega(J(v), w) \quad \forall u, v \in \mathbb{F}.$$

Moreover, given any unit vector $e_1 \in \mathbb{F}$ the basis $\{e_1, e_2\}$ with $e_2 := J(e_1)$ is a symplectic and orthonormal one. This is to say it spans the phase space

$$\mathbb{F} = \mathbb{R} \cdot e_1 \oplus \mathbb{R} \cdot e_2.$$

Let us remark that associated with J , the symplectic space \mathbb{F} is turned into a complex line. To see this, we form the two-dimensional \mathbb{R} -linear space

$$\mathbb{C}^J := \text{span}\{id, J\}.$$

Here $id, J \in Sp(F)$. The operations on \mathbb{C}^J will be defined in a similar way to those on a field of complex variables; J plays the role of the complex unity i in \mathbb{C} . The action of the field \mathbb{C}^J on \mathbb{F} is determined by the action of the linear maps id and J on \mathbb{F} . This turns \mathbb{F} into a complex line.

The symplectic space \mathbb{F} can naturally be enlarged to a three-dimensional, oriented, Euclidean, real linear space

$$\mathbb{E} := \mathbb{F} \oplus \mathbb{R} \cdot J. \quad (10)$$

We extend \langle, \rangle orthogonally to a scalar product \langle, \rangle_E on all of \mathbb{E} for which $\langle J, J \rangle_E = 1$ and fix a metric volume form μ_E on \mathbb{E} . If its restriction to \mathbb{F} is ω , then

$$\mu_E(e_1, e_2, J) = 1. \quad (11)$$

The oriented, Euclidean linear space \mathbb{E} then admits a *cross product*. With these ingredients we turn the real linear space

$$\mathbb{H} := \mathbb{F} \oplus \mathbb{C}^J = \mathbb{E} \oplus \mathbb{R} \cdot id \quad (12)$$

into a skew-field by setting

$$(\lambda_1 e + u_1) \cdot (\lambda_2 e + u_2) := (\lambda_1 \lambda_2 - \langle u_1, u_2 \rangle) \cdot e + \lambda_1 u_2 + \lambda_2 u_1 + u_1 \times u_2 \quad (13)$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and any $u_1, u_2 \in \mathbb{E}$. Here e replaces id . This skew-field is called the *quaternions* \mathbb{H}_F (cf. [6]).

From the matrix structure of J , we deduce

$$J(v) = j \times v = j \cdot v \quad \forall v \in \mathbb{F}. \quad (14)$$

for a *unique* unit vector $j \in \mathbb{E}$, perpendicular to \mathbb{F} ; hence \mathbb{F} is the tangent plane to the unit sphere S^2 in \mathbb{E} at j . As a consequence $j \cdot j = -e$. Because of (14), we identify the vector $j \in \mathbb{E}$ with the map J on \mathbb{F} . Hence, in dealing with \mathbb{H} we will replace J by j in what follows.

By the universal property, \mathbb{H}_F is the *Clifford algebra* of the plane \mathbb{F} endowed with $-\langle, \rangle$ (cf. [7]) and hence we call it the *geometric Clifford algebra* of the symplectic space \mathbb{F} . Occasionally, we express this by the symbol \mathbb{F}^j . Vice versa, by (13) the algebra \mathbb{H}_F determines the bilinear form $-\langle, \rangle$ on \mathbb{F} , because $h^2 = -\langle h, h \rangle$ holds for any $h \in \mathbb{F}$. Moreover, \mathbb{C}^j is a commutative subfield

of \mathbb{H}_F and complements the linear space \mathbb{F} within \mathbb{H}_F and thus the Clifford algebra \mathbb{H}_F is \mathbb{Z}_2 graded.

The space \mathbb{F} is embedded as a phase space if an orthonormal basis $h, k \subset \mathbb{F}$ is chosen. In this case we have the orthonormal basis h, k, j, e in \mathbb{H}_F .

The unit sphere $S^3 \subset \mathbb{H}_F$ is a group, namely $SU(2)$ (cf. [1], of which the two-sphere S^2 in \mathbb{E} is the equator. S^2 constitutes all $j \in \mathbb{E}$ with $j^2 = -e$. Alternately formulated, it constitutes all the almost complex structures J on all two-dimensional linear subspaces in \mathbb{E} .

Clearly ω is algebraically encoded in the Clifford algebra \mathbb{H}_F by (14) and (6), or more sophisticatedly expressed, the Lie bracket of the *Heisenberg algebra* on $\mathbb{E} = \mathbb{F} \oplus \mathbb{R} \cdot j$ given by ω is determined by the product in \mathbb{H}_F (cf. [2]). The metric $-\langle, \rangle_E$, being part of the product in \mathbb{H}_F , plays a central role in special relativity (cf. [2]). More generally, \mathbb{H}_F encodes the Euclidean geometry of the line, plane, three- and four-space, as well as the Minkowski geometry of the plane and four-space.

Finally, given a Clifford algebra in terms of a skew-field of quaternions, \mathbb{H} , then

$$\mathbb{H} = \mathbb{H}_F \tag{15}$$

for some symplectic plane \mathbb{F} . In fact, the tangent plane \mathbb{F} , say, at any $j \in S^2 \subset \mathbb{H}$ is symplectic with symplectic structure ω given by

$$\omega(v, w) = \langle j \cdot v, w \rangle = \langle j, v \cdot w \rangle \quad \forall v, w \in \mathbb{F}. \tag{16}$$

As easily seen, (14) holds true. Thus the symplectic plane \mathbb{F} can be reconstructed from \mathbb{H} ; in this sense \mathbb{H} is characteristic for \mathbb{F} .

In fact the Clifford algebra \mathbb{H} in (15), the skew-field of quaternions encodes all symplectic planes in the three-dimensional real linear space \mathbb{E} , as is easily seen.

As mentioned in section 2.1 the Clifford algebra \mathbb{H} emanates from processes; since the symplectic space \mathbb{F} can be reconstructed from its geometric Clifford algebras by (15), similarly the symplectic space \mathbb{F} can also be reconstructed from process as well.

The construction of the geometric Clifford algebra of a $2n$ -dimensional symplectic space \mathbb{F} is based on the following lemma (cf. [6]):

Lemma 2 *Let \mathbb{F} be a $2n$ -dimensional symplectic space with ω as (constant) symplectic form. The linear space can be decomposed into the direct sum*

$$\mathbb{F} = \bigoplus_s \mathbb{F}_s \tag{17}$$

(where s runs from 1 to n) which admits a basis $\{e_1, \dots, e_{2n}\}$ such that the intersection $\{e_1, \dots, e_{2n}\} \cap \mathbb{F}_s$ is a symplectic basis of ω_s , the restriction of ω to \mathbb{F}_s for each s . In addition there is a unique scalar product \langle, \rangle_s for each s , such that ω_s is represented uniquely by J_s having (7) as its matrix and which satisfies (8).

Hence there is a unique scalar product \langle, \rangle on $\mathbb{F} = \bigoplus_s \mathbb{F}_s$ yielding \langle, \rangle_s in lemma 2 if restricted to \mathbb{F}_s for each s . From lemma 1 each \mathbb{F}_s in (17) is a phase space. If \mathbb{H}_{F_s} denotes the Clifford algebra of $-\langle, \rangle_s$ for all s , the Clifford algebra \mathbb{H}_F of $-\langle, \rangle$ on \mathbb{F} is given by the 2^{2n} -dimensional algebra

$$\mathbb{H}_F = \bigotimes_s \mathbb{H}_{F_s}. \quad (18)$$

Here s runs from 1 to n and \bigotimes denotes the *graded tensor product* (cf. [7]. This tensor product is called the anticommutative tensor product) of the \mathbb{Z}_2 graded algebras \mathbb{H}_{F_s} . Again here we will call it the *geometric Clifford algebra* of the symplectic space F ; obviously \mathbb{H}_F is *characteristic* for \mathbb{F} . Clearly each \mathbb{H}_{F_s} is a subalgebra of \mathbb{H}_F .

As an example, consider m particles moving in \mathbb{R}^3 endowed with the canonical scalar product and its natural orthonormal basis. The configuration space is the m -fold direct sum $\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3$ which is isomorphic to the m -fold Cartesian product of \mathbb{R}^3 with itself, a $3m$ -dimensional linear space. Let the q -coordinates be enumerated from q_1 to q_{3m} . The phase space is isomorphic to $(\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3) \times (\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3)$, a $2 \cdot 3m$ -dimensional linear space. The first factor of this cartesian product constitutes all possible spatial positions of the m particles, while the second factor contains all the momenta. This phase space together with the canonical symplectic form ω , say, is a $2n$ dimensional symplectic space. Here $n = 3m$. Now we regroup the coordinates in this phase space. We take the plane \mathbb{F}_1 generated by the first spatial coordinate q_1 and its moment p_1 together with the canonical symplectic form. Passing to the second spatial coordinates and its momenta yields \mathbb{F}_2 etc. In this way we obtain

$$\mathbb{F} := \bigoplus_s \mathbb{F}_s \cong (\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3) \times (\mathbb{R}^3 \oplus \dots \oplus \mathbb{R}^3), \quad (19)$$

and endow it with the natural scalar product. Here s varies from 1 to $3m$. The symplectic bases of \mathbb{F} in which the canonical symplectic form ω decomposes according to lemma 2 is identical with the canonical orthonormal basis of \mathbb{F} in (19).

3.2 The Clifford Algebra determined by $sp(F)$

Another realization of the geometric Clifford algebra of the *two-dimensional symplectic space* \mathbb{F} is obtained by means of $sp(F)$, the Lie algebra of the symplectic group $Sp(F)$. At first we will construct a natural splitting of $sp(F)$ and a symplectic structure on a two-dimensional subspace. The basis of this construction is $\text{End } F$, the four-dimensional \mathbb{R} -linear space of all (\mathbb{R} -linear) endomorphisms of \mathbb{F} . This space is split into

$$\text{End } F = \mathbb{R} \cdot id_F \oplus sp(F) \quad (20)$$

where $sp(F)$ is the three-dimensional \mathbb{R} -linear space of all *traceless endomorphisms* of \mathbb{F} . In fact, this splitting is orthogonal with respect to the natural scalar product $\langle, \rangle_{\text{End } F}$, say, defined by

$$\langle A, B \rangle_{\text{End } F} := \frac{1}{2} tr A \circ \tilde{B} \quad \forall A, B \in \text{End } F. \quad (21)$$

Here \tilde{B} is the adjoint of B . In fact the splitting (20) carries a natural *Heisenberg algebra structure* (cf. [2]).

By means of this scalar product we observe that any almost complex structure on a two-dimensional linear subspace of $\text{sp}(F)$ is a unit vector and vice versa any vector in the unit sphere $S^2 \subset \text{sp}(F)$ is an almost complex structure on the tangent plane of S^2 at that vector (cf. 16).

Moreover, let $\mu_{\text{sp}(F)}$ be a constant metric volume form, i.e. a determinant function (cf. [6]) on $\text{sp}(F)$, of the scalar product $\langle A, B \rangle_{\text{sp}(F)}$ in (21).

For a given J with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the linear space $\text{sp}(F)$ decomposes into

$$\text{sp}(F) = \mathbb{R} \cdot J \oplus \Sigma$$

where the matrix of an element in Σ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, obviously a *symmetric matrix* with vanishing trace. (These matrices are formed with respect to an orthonormal basis in F). This yields

$$\text{End}F = \mathbb{R} \cdot id + \mathbb{R} \cdot J + \Sigma$$

and hence

$$\Sigma = \text{span} \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid a, c \in \mathbb{R} \right\}.$$

(Thus $\text{sp}(F)$ is not closed under composition, because of (5)). Also because of (5), the linear two-dimensional space Σ inherits a *symplectic structure* ω_Σ defined by

$$\omega_\Sigma(A, B) := \langle \frac{1}{2} \cdot [A, B], J \rangle_{\text{End}F} \quad \forall A, B \in \text{sp}(F).$$

Since $[A, B] \in \mathbb{R} \cdot J$ for A, B in Σ with $\|A\| = \|B\| = 1$

$$\mu_{\text{sp}(F)}(A, B, J) = \omega_\Sigma(A, B),$$

and hence

$$\mu_{\text{sp}(F)}(A, B, J) = \langle \frac{1}{2} \cdot [A, B], J \rangle_{\text{End}F}.$$

This identifies $\frac{1}{2} \cdot [A, B]$ as the *cross product* of $A, B \in \Sigma$ in $\text{sp}(F)$.

Therefore, by (13) the four-dimensional linear space $\text{End}F$ is turned into a skew-field $\mathbb{H}_{\text{sp}(F)}$ isomorphic to the quaternions, i.e. isomorphic to the geometric Clifford algebra \mathbb{H}_F of the symplectic space \mathbb{F} . (Note $\text{End}F$ is a Clifford algebra with the composition as multiplication, as mentioned in section 2.2. It is not isomorphic to $\mathbb{H}_{\text{sp}(F)}$).

We point out that Σ and ω_Σ can be naturally identified with \mathbb{F} and ω respectively as shown in (cf. [2]). Hence

$$\mathbb{H}_F \equiv \mathbb{H}_{\text{sp}(F)}. \tag{22}$$

In the following we will use either picture.

3.3 Poisson Algebras of the Phase Space

In order to derive a characteristic Clifford algebra from the Poisson algebra of all *homogenous quadratic polynomials* on a two-dimensional phase space \mathbb{F} , we will need to study the Poisson algebra of \mathbb{F} .

Our first goal is to establish a natural isomorphism between the Poisson algebra \mathcal{Q} of all *homogeneous quadratic polynomials* on the *two-dimensional phase space* \mathbb{F} (cf. [14]) and $sp(\mathbb{F})$ (cf. [16]). The key to this isomorphism will be the construction of *Hamiltonian vector fields* of homogenous quadratic polynomials (cf. [8]).

The calculation in what follows can easily be visualized if we use the quaternions \mathbb{H}_F of the phase space \mathbb{F} . This skew-field contains \mathbb{F} as well as the line $\mathbb{R} \cdot j$ which is perpendicular to \mathbb{F} .

The *coordinate system* in \mathbb{F} determined by the orthonormal unit vectors e_q and $e_p := j \cdot e_q$ respectively, of the (linear) *coordinate functions* f_q and f_p on \mathbb{F} . This allows us to introduce the concept of a polynomial on \mathbb{F} in *two variables* q and p , say. The collection \mathcal{Q} of all homogenous quadratic polynomials on \mathbb{F} obviously forms an \mathbb{R} -vector space.

Given a polynomial pol on \mathbb{F} , let $f_{pol} : \mathbb{F} \rightarrow \mathbb{R}$ be its polynomial function in the variables q and p . In the following we will identify the collection

$$P_{\mathcal{Q}} := \{f_{pol} | pol \in \mathcal{Q}\}$$

with \mathcal{Q} by identifying any pol with f_{pol} , if no confusion arises.

We begin our investigations of \mathcal{Q} by the geometric study of the notion of Hamiltonian vector fields on the two-dimensional phase space. Let $C^\infty(F, \mathbb{R})$ be the collection of all smooth \mathbb{R} -valued functions of \mathbb{F} . It is an \mathbb{R} -algebra under pointwisely defined operations. Given $f \in C^\infty(\mathbb{F}, \mathbb{R})$, its *Hamiltonian vector field* X_f is defined by

$$\omega(X_f, X) = df(X) \tag{23}$$

for any smooth vector field X on F (cf. [14]). Clearly, X_f is smooth as well.

Throughout this paper, we will replace the vector field $X_f = (\mathbf{a}_f, id)$ by its principal part \mathbf{a}_f , mapping \mathbb{F} into itself. Thus (23) turns into

$$\omega(\mathbf{a}_f(h), k) = df(h)(k) \quad \forall h, k \in \mathbb{F}. \tag{24}$$

Here $df(h)(k)$ is the derivative of f at h evaluated at $k \in \mathbb{F}$.

It follows from (9) and (16) that equation (24) can be rewritten in terms of the scalar product, yielding

$$\omega(\mathbf{a}_f(h), k) = \langle j \times \mathbf{a}_f(h), k \rangle = df(h)(k) \quad \forall h, k \in \mathbb{F}$$

which by (14) yields in turn a link to $\nabla f \in C^\infty(F, \mathbb{R})$, namely that the principal part \mathbf{a}_f of the Hamiltonian vector field which is

$$\mathbf{a}_f(h) = -j \times \nabla f(h) = -j \cdot \nabla f(h) \quad \forall h \in \mathbb{F} \tag{25}$$

where \cdot denotes the multiplication in \mathbb{H}_F .

A direct calculation in the orthonormal basis given by the unit vectors e_q and e_p shows:

$$\mathbf{a}_f(h) = \frac{\partial f(h)}{\partial p} \cdot e_q - \frac{\partial f(h)}{\partial q} \cdot e_p \quad \forall h \in \mathbb{F}. \quad (26)$$

As an example, by (25) the Hamiltonian vector fields of the coordinate functions f_q and f_p obviously are

$$\mathbf{a}_{f_q} = -j \cdot e_q = -e_p \quad \text{and} \quad \mathbf{a}_{f_p} = -j \cdot e_p = e_q. \quad (27)$$

Clearly, $j = e_q \cdot e_p$ holds true.

We study the above notions by turning to $\mathcal{Q} \subset C^\infty(F, \mathbb{R})$. Of special interest with respect to our goal is the linear dependence of $\mathbf{a}_f(h)$ on the position $h \in \mathbb{F}$ for any $f \in \mathcal{Q}$. By (24) this means that

$$\mathbf{a}_f \in \text{End}F; \quad (28)$$

hence both sides of (24) depend bilinearly on h and k for any $f \in \mathcal{Q}$.

The quality of \mathbf{a}_f will become clear if we notice that \mathcal{Q} is generated by $\frac{q^2}{2}, \frac{p^2}{2}$ and $q \cdot p$.

By (26) the Hamiltonian vector fields of $f_{\frac{q^2}{2}}, f_{\frac{p^2}{2}}$ and $f_{q \cdot p}$ are the following for all $h \in F$ of the form $h = q \cdot e_q + p \cdot e_p$:

$$\mathbf{a}_{f_{\frac{q^2}{2}}}(h) = -q \cdot e_p \quad \mathbf{a}_{f_{\frac{p^2}{2}}}(h) = p \cdot e_q \quad \mathbf{a}_{f_{q \cdot p}}(h) = q \cdot e_q - p \cdot e_p. \quad (29)$$

Their respective matrices $M(\mathbf{a}_{f_{q^2}}), M(\mathbf{a}_{f_{p^2}})$ and $M(\mathbf{a}_{f_{q \cdot p}})$ formed with respect to e_q and e_p (cf. 28) are

$$M(\mathbf{a}_{f_{q^2}}) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad M(\mathbf{a}_{f_{p^2}}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M(\mathbf{a}_{f_{q \cdot p}}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (30)$$

These matrices are all *tracefree* which means they belong to the Lie algebra $\text{sp}(F)$.

Since the endomorphisms $\mathbf{a}_{f_{q^2}}, \mathbf{a}_{f_{p^2}}$ and $\mathbf{a}_{f_{q \cdot p}}$ of \mathbb{F} generate $\text{sp}(F)$, we have a surjective linear map

$$\begin{aligned} \text{ham} : \mathcal{Q} &\rightarrow \text{sp}(F) \\ \text{ham}(f) &:= \mathbf{a}_f \quad \forall f \in \mathcal{Q}. \end{aligned} \quad (31)$$

ham is a *linear isomorphism* since both the domain and the range are three-dimensional.

Next we will verify that \mathcal{Q} forms a Poisson sub-algebra of $C^\infty(F, \mathbb{C})$. To this end, we define a *Poisson bracket* on $C^\infty(F, \mathbb{R})$ and show that $C^\infty(F, \mathbb{R})$ is a Poisson algebra (cf. [4]), which means that the \mathbb{R} -linear space, $C^\infty(F, \mathbb{R})$, has a natural Lie algebra structure. Therefore, let $f, g \in C^\infty(F, \mathbb{R})$ with \mathbf{a}_f and \mathbf{a}_g being the principal parts of their Hamiltonian vector fields. Obviously,

$$\omega(\mathbf{a}_f(h), \mathbf{a}_g(h)) = df(h)(\mathbf{a}_g(h)).$$

Thus

$$df(\mathbf{a}_g) = \langle j, \mathbf{a}_f \times \mathbf{a}_g \rangle.$$

To evaluate the cross product we use equation (26) to obtain

$$\mathbf{a}_f \times \mathbf{a}_g = -\frac{\partial f}{\partial p} \frac{\partial g}{\partial q} \cdot e_q \times e_p - \frac{\partial f}{\partial q} \cdot \frac{\partial g}{\partial p} \cdot e_p \times e_q$$

and, therefore,

$$\mathbf{a}_f \times \mathbf{a}_g = -\left(\frac{\partial f}{\partial q} \cdot \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} \right) \cdot j. \quad (32)$$

Defining the *Poisson bracket* of f and g in $C^\infty(F, \mathbb{R})$ by

$$\{f, g\} := \frac{\partial f}{\partial q} \cdot \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q}, \quad (33)$$

yields

$$\{f, g\} \cdot j = -\mathbf{a}_f \times \mathbf{a}_g; \quad (34)$$

hence this bracket can be expressed as

$$\{f, g\} \cdot e = j \cdot (\mathbf{a}_f \times \mathbf{a}_g) \quad \forall f, g \in C^\infty(F, \mathbb{R}).$$

They satisfy a Jacobi identity, i.e.

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0.$$

Therefore, $C^\infty(F, \mathbb{R})$ is a Lie algebra under the Poisson bracket (33) (cf. [16]).

Since $\{f, g\} \in C^\infty(F, \mathbb{R})$, we may investigate $\mathbf{a}_{\{f, g\}}$. Let $[\mathbf{a}_f, \mathbf{a}_g]$ denote the principal part of the *Lie bracket* $[X_f, X_g]$ defined by

$$[\mathbf{a}_f, \mathbf{a}_g] := d\mathbf{a}_g(\mathbf{a}_f) - d\mathbf{a}_f(\mathbf{a}_g).$$

As is well known

$$d\{f, g\}(h) = -\omega^a([\mathbf{a}_f, \mathbf{a}_g], h) \quad \forall h \in \mathbb{F}$$

showing

$$\mathbf{a}_{\{f, g\}} = -[\mathbf{a}_f, \mathbf{a}_g] \quad \forall f, g \in C^\infty(F, \mathbb{R}). \quad (35)$$

Denoting by $\text{Ham } \mathbb{F}$ the collection of all Hamiltonian vector fields on \mathbb{F} , the linear isomorphism in (31) extends to the linear map

$$\text{ham} : C^\infty(F, \mathbb{R}) \rightarrow \text{Ham } \mathbb{F} \quad (36)$$

determined by

$$\text{ham } f := \mathbf{a}_f \quad \forall f \in C^\infty(F, \mathbb{R})$$

which satisfies (35).

Now focussing on \mathcal{Q} again, the Poisson brackets $\{f_{\frac{q^2}{2}}, f_{\frac{p^2}{2}}\}$, $\{f_{q \cdot p}, f_{\frac{q^2}{2}}\}$ and $\{f_{q \cdot p}, f_{\frac{p^2}{2}}\}$ are

$$\{f_{\frac{q^2}{2}}, f_{\frac{p^2}{2}}\} = f_{q \cdot p} \quad \{f_{q \cdot p}, f_{\frac{q^2}{2}}\} = f_{q^2} \quad \{f_{q \cdot p}, f_{\frac{p^2}{2}}\} = f_{p^2} \quad (37)$$

which are easily calculated either from (33) or (34). Hence we have shown (cf. [8])

Proposition 3 \mathcal{Q} closes under the Poisson brackets and is a Poisson sub-algebra of $C^\infty(\mathbb{F}, \mathbb{R})$.

The formalism we have used here to introduce Hamiltonian vector fields and Poisson brackets is adapted to our setting for two-dimensional phase spaces. For a general formalism we refer to [14] or [8].

3.4 Clifford Algebra Structure based on \mathcal{Q} .

Next, we extend the Poisson algebra of all homogenous quadratic polynomials on a *two-dimensional* phase space to a skew-field of quaternions, a Clifford algebra and generalize this to *2n-dimensional* symplectic space.

Because of (22) we will now relate the Lie algebra $\mathfrak{sl}(F)$ of $\mathrm{SL}(F)$, the special linear group of \mathbb{F} , to \mathcal{Q} (cf. subsection 3.2). From (28) we have

$$\mathfrak{sp}(F) = \mathfrak{sl}(F). \quad (38)$$

According to (26), for any $f \in \mathcal{Q}$ the trace free matrix $M(\mathbf{a}_f)$ of \mathbf{a}_f is

$$M(\mathbf{a}_f) = \begin{pmatrix} \frac{\partial f}{\partial p}(e_q) & \frac{\partial f}{\partial p}(e_p) \\ -\frac{\partial f}{\partial q}(e_q) & -\frac{\partial f}{\partial q}(e_p) \end{pmatrix}. \quad (39)$$

Moreover, a direct calculation provides us with the key result on *ham* in (31) for the construction of the Clifford algebra containing \mathcal{Q} :

Proposition 4

$$\mathit{ham} : \mathcal{Q} \rightarrow \mathfrak{sp}(F) \quad (40)$$

is a Lie algebra isomorphism satisfying

$$\mathit{ham}\{f, g\} = [\mathbf{a}_f, \mathbf{a}_g]_{\mathfrak{sl}(F^a)} \quad \forall f, g \in \mathcal{Q}. \quad (41)$$

Here

$$[\mathbf{a}_f, \mathbf{a}_g]_{\mathfrak{sl}(F^a)} := \mathbf{a}_f \circ \mathbf{a}_g - \mathbf{a}_g \circ \mathbf{a}_f \quad \forall f, g \in \mathcal{Q}$$

is the commutator in $\mathfrak{sl}(F^a)$.

Now we enlarge \mathcal{Q} to $\mathbb{H}_{\mathcal{Q}} := \mathcal{Q} \oplus \mathbb{R} \cdot e$ and extend *ham* in proposition 4 to

$$\mathit{ham} : \mathbb{H}_{\mathcal{Q}} \rightarrow \mathrm{End}F \quad (42)$$

by sending e to the identity *id*. This is a linear isomorphism. As shown in section 2.2 the linear space $\mathrm{End}F$ carries a natural Clifford algebra structure arising from the Lie algebra $\mathfrak{sp}(F)$. This structure can be pulled back to $\mathbb{H}_{\mathcal{Q}}$ by *ham*. The resulting skew-field is called here the *Poisson Clifford algebra* of the phase space and is also denoted by $\mathbb{H}_{\mathcal{Q}}$. Compared with the defining equation (13) for the product in \mathbb{H}_F , the Poisson bracket on \mathcal{Q} serves as the cross product. By construction we have

$$\mathbb{H}_{\mathbb{F}} = \mathbb{H}_{\mathfrak{sp}(F)} \cong \mathbb{H}_{\mathcal{Q}} \quad (43)$$

as Clifford algebras.

In the case where the phase space \mathbb{F} is *2n-dimensional*, we proceed as in section 3.1. We decompose \mathbb{F} into $\mathbb{F} = \oplus_s \mathbb{F}_s$ and form for each s the Clifford algebra $\mathbb{H}_{\mathcal{Q}_s}$.

Hence $\otimes_s \mathbb{H}_{\mathcal{Q}_s}$ is a Clifford algebra, called here the *Poisson Clifford algebra* of the phase space F . Obviously

$$\mathbb{H}_F = \otimes_s \mathbb{H}_s \cong \otimes_s \mathbb{H}_{\mathcal{Q}_s} \quad (44)$$

holds true.

\mathcal{Q} , the collection of all functions of homogenous quadratic polynomials of \mathbb{F} , is the image of a natural subspace of the Poisson Clifford algebra $\otimes_s \mathbb{H}_s$, as seen as follows: Based on (29) we restrict *ham* in (36) to \mathcal{Q} and calculate in $\otimes_s \mathbb{H}_{sp(F_s)}$. The tensor $q_s \cdot p_r e \otimes \dots \otimes e_s \otimes \dots \otimes e_r \dots \otimes e$ in $\otimes_s \mathbb{H}_{sp(F_s)}$ applied to an element of $\mathbb{F} = \oplus_s \mathbb{F}_s$ in (17) (via the canonical scalar product) yields $q_s \cdot p_r$ for all $s, r \leq n$. Thus both $q_s \cdot p_r e \otimes \dots \otimes e_s \otimes \dots \otimes e_r \dots \otimes e$ as well as $q_s \cdot p_r e \otimes \dots \otimes e_r \otimes \dots \otimes e_s \dots \otimes e$ map linearly to $q_s \cdot p_r$ in \mathcal{Q} for all $s, r \leq n$. In this way we obtain all of \mathcal{Q} .

3.5 A realization of \mathbb{H}_F consisting of Operators

We close our studies of characteristic Clifford algebras of symplectic spaces by introducing for any finite-dimensional phase space \mathbb{F} a realization of \mathbb{H}_F in terms of a Clifford algebra consisting of *Hermitian operators*. This will allow us to associate eigen-values to any element of \mathbb{H}_F . (For an alternative approach, consult [9]).

This realization is constructed first for a two-dimensional phase space \mathbb{F} and then extended to $2n$ -dimensions $\mathbb{F} = \oplus_s \mathbb{F}_s$ by means of (18).

Given a two-dimensional phase space \mathbb{F} , this construction is based on the *quantization map*

$$Q : \mathcal{Q} \rightarrow \text{End}(L^2(\mathbb{R}, \mathbb{C}))$$

(cf. [2]). It is defined to be the composition of

$$\mathcal{Q} \xrightarrow{\text{ham}} sp(F) \xrightarrow{-i \cdot dU_{Mp}} \text{End}(L^2(\mathbb{R}, \mathbb{C})), \quad (45)$$

i.e.

$$Q := -i \cdot dU_{Mp} \circ \text{ham}. \quad (46)$$

The right hand composite is made up by the (injective) infinitesimal metaplectic representation dU_{Mp} (cf. [2] or [4]) multiplied by the factor $-i$ and the Lie algebra isomorphism *ham* in (31). The values of dU_{Mp} are *skew-symmetric operators* acting on the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$, hence $\mp i \cdot dU_{Mp}$ maps into a collection of Hermitian operators. The choice of the factor $-i$ in Q (cf. (46)) was made to guarantee that

$$Q(\{f, g\}) = [Q(f), Q(g)] \quad (47)$$

holds true for any pair of polynomial function $f, g \in \mathcal{Q}$. Thus Q *quantizes* any homogenous quadratic polynomial defined on \mathbb{F} . (The linear isomorphism *ham* in (31) can be defined for any finite-dimensional phase space \mathbb{F} (cf. [8])).

The isomorphism (46) obviously extends to a Clifford algebra isomorphism also denoted by *ham*. This is to say we have

$$\text{ham} : \mathbb{H}_{\mathcal{Q}} \rightarrow \mathbb{H}_{sp(F)}, \quad (48)$$

mapping $e \in \mathbb{H}_{\mathcal{Q}}$ to *id*.

We extend the monomorphism dU_{Mp} to all $\text{End } F = \mathbb{R} \cdot id_F \oplus sp(F)$ as a linear injective map (still called dU_{Mp})

$$dU_{Mp} : \mathbb{R} \cdot id_F \oplus sp(F) \rightarrow \text{End}(L^2(\mathbb{R}, \mathbb{C}))$$

by setting

$$dU_{Mp}(id_F) = id_{\text{End}(L^2(\mathbb{R}, \mathbb{C}))}.$$

Together with (46), this means we have a linear map

$$Q : \mathbb{H}_F \rightarrow \text{im } Q.$$

Here $\text{im } Q$ is the (four-dimensional) image of Q .

Pushing forward by Q , the skew-field structure of \mathbb{H}_F to $\text{im } Q$ yields the structure of a skew-field of quaternions on $\text{im } Q$, a Clifford algebra as mentioned above. It is denoted by \mathbb{H}_Q .

To extend the quantization map Q to $\mathbb{H}_F = \otimes_s \mathbb{H}_s$ for an *2n-dimensional* phase space \mathbb{F} as in (18), we form the quantization map for each s yielding Q_s of which the image is $\text{im } Q_s$ and set

$$Q := \otimes_s Q_s. \quad (49)$$

The image of Q is the graded tensor product

$$\otimes_s \text{im } Q_s,$$

a finite dimensional Clifford algebra, referd to as the *realization* of \mathbb{H}_F by means of *Hermitian operators*.

4 Acknowledgement

We are deeply indebted to the generosity of Georg Wikman in sponsoring the Askloster Seminars where different aspects of this work were presented and discussed. Without those meetings the work presented in this paper would not have been possible.

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